Lecture (08)

Finding prim Implicates, essential prime Implicates, and minimum products of minterms expression

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consensus theorem

$$XY + \overline{X}Z + YZ$$

$$= XY + \overline{X}Z + YZ \cdot (X + \overline{X})$$

$$= XY + \overline{X}Z + XYZ + \overline{X}YZ$$

$$= XY(1 + Z) + \overline{X}Z(1 + Y)$$

$$= XY + \overline{X}Z$$

$$\therefore XY + \overline{X}Z + YZ = XY + \overline{X}Z$$

Determination of Prime Implicates

- the function must be given as a sum of minterms.
- all of the prime implicants of a function are systematically formed by combining minterms

→ To reduce the required number of comparisons, the binary minterms are sorted into groups according to the number of 1's in each term

$$f(a, b, c, d) = \Sigma\, m(0, 1, 2, 5, 6, 7, 8, 9, 10, 14)$$

is represented by the following list of minterms:

group 0	0	0000
group 1	1	0001
	2	0010
	8	1000
group 2	5	0101
	6	0110
	9	1001
	10	1010
group 3	7	0111
	14	1110

- Two terms in any two groups can be combined as they differ in exactly one variable. $XY + XY' = X.$
- First, we will compare the term in group 0 with all of the terms in group 1.
- Terms 0000 and 0001 can be combined to eliminate the fourth variable, which yields 000–.
- Similarly, 0 and 2 combine to form 00–0 (*a*′*b*′*d*′), and 0 and 8 combine to form –000 (*b*′*c*′*d*′). The resulting terms are listed in Column II
- the corresponding decimal numbers differ by a power of 2 (1, 2, 4, 8, etc.).
- A term may be used more than once because *X* + *X* = *X*.

	Column I	Column II	Column III
group 0	0 0000 ✓	0, 1 000– ✓	0, 1, 8, 9 –00–
group 1	1 0001 ✓	0, 2 00–0 ✓	0, 2, 8, 10 –0–0
	2 0010 ✓	0, 8 –000 ✓	~~0, 8, 1, 9 –00–~~
	8 1000 ✓	1, 5 0–01	~~0, 8, 2, 10 –0–0~~
group 2	5 0101 ✓	1, 9 –001 ✓	2, 6, 10, 14 -- 10
	6 0110 ✓	2, 6 0–10 ✓	~~2, 10, 6, 14 -- 10~~
	9 1001 ✓	2, 10 –010 ✓	
	10 1010 ✓	8, 9 100– ✓	
group 3	7 0111 ✓	8, 10 10–0 ✓	
	14 1110 ✓	5, 7 01–1	
		6, 7 011–	
		6, 14 –110 ✓	
		10, 14 1–10 ✓	

- Note that the terms in Column II have been divided into groups, according to the number of 1’s in each term
- Again, we apply *XY* + *XY*′ = *X* to combine pairs of terms in Column II.
- In order to combine two terms, the terms must have the same variables, and the terms must differ in exactly one of these variables

	Column I			Column II			Column III	
group 0	0	0000 ✓		0, 1	000– ✓		0, 1, 8, 9	–00–
group 1	1	0001 ✓		0, 2	00–0 ✓		0, 2, 8, 10	–0–0
	2	0010 ✓		0, 8	–000 ✓		~~0, 8, 1, 9~~	~~–00–~~
	8	1000 ✓		1, 5	0–01		~~0, 8, 2, 10~~	~~–0–0~~
group 2	5	0101 ✓		1, 9	–001 ✓		2, 6, 10, 14	-- 10
	6	0110 ✓		2, 6	0–10 ✓		~~2, 10, 6, 14~~	~~-- 10~~
	9	1001 ✓		2, 10	–010 ✓			
	10	1010 ✓		8, 9	100– ✓			
group 3	7	0111 ✓		8, 10	10–0 ✓			
	14	1110 ✓		5, 7	01–1			
				6, 7	011–			
				6, 14	–110 ✓			
				10, 14	1–10 ✓			

- Note that there are three pairs of duplicate terms in Column III.
- After deleting the duplicate terms, we compare terms from the two groups in Column III.

	Column I	Column II	Column III
group 0	0 0000 ✓	0, 1 000– ✓	0, 1, 8, 9 –00–
group 1	1 0001 ✓	0, 2 00–0 ✓	0, 2, 8, 10 –0–0
	2 0010 ✓	0, 8 –000 ✓	~~0, 8, 1, 9 –00–~~
	8 1000 ✓	1, 5 0–01	~~0, 8, 2, 10 –0–0~~
group 2	5 0101 ✓	1, 9 –001 ✓	2, 6, 10, 14 -- 10
	6 0110 ✓	2, 6 0–10 ✓	~~2, 10, 6, 14 -- 10~~
	9 1001 ✓	2, 10 –010 ✓	
	10 1010 ✓	8, 9 100– ✓	
group 3	7 0111 ✓	8, 10 10–0 ✓	
	14 1110 ✓	5, 7 01–1	
		6, 7 011–	
		6, 14 –110 ✓	
		10, 14 1–10 ✓	

- The terms which have not been checked off because they cannot be combined with other terms are called prime implicants.

	Column I	Column II	Column III
group 0	0 0000 ✓	0, 1 000– ✓	0, 1, 8, 9 –00–
group 1	1 0001 ✓	0, 2 00–0 ✓	0, 2, 8, 10 –0–0
	2 0010 ✓	0, 8 –000 ✓	~~0, 8, 1, 9 –00–~~
	8 1000 ✓	1, 5 0–01	~~0, 8, 2, 10 –0–0~~
group 2	5 0101 ✓	1, 9 –001 ✓	2, 6, 10, 14 -- 10
	6 0110 ✓	2, 6 0–10 ✓	~~2, 10, 6, 14 -- 10~~
	9 1001 ✓	2, 10 –010 ✓	
	10 1010 ✓	8, 9 100– ✓	
group 3	7 0111 ✓	8, 10 10–0 ✓	
	14 1110 ✓	5, 7 01–1	
		6, 7 011–	
		6, 14 –110 ✓	
		10, 14 1–10 ✓	

- As the function is equal to the sum of its prime implicants.

$$f = a'c'd + a'bd + a'bc + b'c' + b'd' + cd'$$

(1, 5) (5, 7) (6, 7) (0, 1, 8, 9) (0, 2, 8, 10) (2, 6, 10, 14)

- Using the consensus theorem to eliminate redundant terms yields

$$f = a'c'd + a'bd + a'bc + b'c' + d'b' + cd'$$

$$a'c'd + a'bd + a'bc$$

$$a'(c'd + bd + bc)$$

$$a'(c'd + bc)$$

$$a'c'd + a'bc$$

$$b'c' + cd'$$

$$\downarrow$$

$$b'c' + cd'$$

- *implicant term:*

Given a function *F* of *n* variables, a product term *P* is an *implicant* of *F* iff for every combination of values of the *n* variables for which *P* = 1, *F* is also equal to 1.

$$F(a,b,c) = a'b'c' + ab'c' + ab'c + abc$$

$$b'c'(a'+a) + ac(b'+b)$$

$$b'c \quad + \quad ac$$

- *If* $a'b'c' = 1$, then $F = 1$; if $ac = 1$, then $F = 1$; etc. Hence, the terms $a'b'c'$, ac, etc., are implicants of F.
- bc is *not* an implicant of F because when $a = 0$ and $b = c = 1$, $bc = 1$ and $F = 0$.
- Every minterm of F is also an implicant of F, and so
- is any term formed by combining two or more minterms.

- *prime implicant*

A *prime implicant* of a function *F* is a product term implicant which is no longer an implicant if any literal is deleted from it.

$$F(a,b,c) = a'b'c' + ab'c' + ab'c + abc$$

$$b'c'(a'+a) + ac(b'+b)$$

$$b'c + ac$$

- the implicant *a′b′c′ is not* a *prime* implicant because *a′* can be eliminated, and the resulting term (*b′c′*) is still an implicant of *F*
- The implicants *b′c′* and *ac* are *prime implicants* because *if* we delete a literal from either term, the term will no longer be an implicant of *F*

The Prime Implicant Chart

- The minterms of the function are listed across the top of the chart, and the prime implicants are listed down the side.
- If a prime implicant covers a given minterm, an X is placed at the intersection of the corresponding row and column.
- If a minterm is covered by only one prime implicant, then that prime implicant is called an *essential* prime implicant

- $f = a'c'd + a'bd + a'bc + b'c' + b'd' + cd'$

  $(1, 5)\quad (5, 7)\quad (6, 7)\quad (0, 1, 8, 9)\quad (0, 2, 8, 10)\quad (2, 6, 10, 14)$

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	b′c′	×	×					×	⊗		
(0, 2, 8, 10)	b′d′	×		×				×		×	
(2, 6, 10, 14)	cd′			×		×				×	⊗
(1, 5)	a′c′d		×		×						
(5, 7)	a′bd				×		×				
(6, 7)	a′bc					×	×				

- A minimum set of prime implicants must now be chosen to cover the remaining columns

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	×	×					×	⊗		
(0, 2, 8, 10)	$b'd'$	×		×				×		×	
(2, 6, 10, 14)	cd'			×		×				×	⊗
(1, 5)	$a'c'd$		×		×						
(5, 7)	$a'bd$				×		×				
(6, 7)	$a'bc$					×	×				

$f = a'c'd + a'bc + b'c' + cd'$

- A minimum set of prime implicants must now be chosen to cover the remaining columns

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	×	×					×	⊗		
(0, 2, 8, 10)	$b'd'$	×		×				×		×	
(2, 6, 10, 14)	cd'			×		×				×	⊗
(1, 5)	$a'c'd$		×		×						
(5, 7)	$a'bd$				×		×				
(6, 7)	$a'bc$					×	×				

$$f = b'c' + cd' + a'bd$$

Example

- A prime implicant chart which has two or more X's in every column is called a *cyclic*
- prime implicant chart. The following function has such a chart

$$F = \Sigma\, m(0, 1, 2, 5, 6, 7)$$

$$F = \Sigma\, m(0, 1, 2, 5, 6, 7)$$

- Derivation of prime implicants:

0	000 ✓	0, 1	00–		$a'b'$
1	001 ✓	0, 2	0–0		$a'c'$
2	010 ✓	1, 5	–01		$b'c$
5	101 ✓	2, 6	–10		bc'
6	110 ✓	5, 7	1–1		ac
7	111 ✓	6, 7	11–		ab

- Build prime implicant chart, to find minimum products

			0	1	2	5	6	7
① →	(0, 1)	$a'b'$	×	×				
	(0, 2)	$a'c'$	×		×			
	(1, 5)	$b'c$		×		×		
② →	(2, 6)	bc'			×		×	
③ →	(5, 7)	ac				×		×
	(6, 7)	ab					×	×

$$F = a'b' + bc' + ac.$$

Petrick's Method

- Petrick's method is a technique for determining all minimum sum-of-products solutions from a prime implicant chart
- we will label the rows of the table *P*1, *P*2, *P*3, etc.

$$F = \Sigma\, m(0, 1, 2, 5, 6, 7)$$

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	×	×				
P_2	(0, 2)	$a'c'$	×		×			
P_3	(1, 5)	$b'c$		×		×		
P_4	(2 6)	bc'			×		×	
P_5	(5, 7)	ac				×		×
P_6	(6, 7)	ab					×	×

- Minterm 0 has X's in rows *P*1 and *P*2, we must choose row *P*1 or *P*2 in order to cover minterm 0. Therefore, the expression (*P*1 + *P*2) must be true.
- In order to cover minterm 1, we must choose row *P*1 or *P*3; therefore, (*P*1 + *P*3) must be true.

			0	1	2	5	6	7
P_1	(0, 1)	*a′b′*	×	×				
P_2	(0, 2)	*a′c′*	×		×			
P_3	(1, 5)	*b′c*		×		×		
P_4	(2 6)	*bc′*			×		×	
P_5	(5, 7)	*ac*				×		×
P_6	(6, 7)	*ab*					×	×

- minterm 2, (*P*2 + *P*4) must be true.
- Similarly, in order to cover minterms 5, 6, and 7, the expressions (*P*3 + *P*5), (*P*4 + *P*6) and (*P*5 + *P*6) must be true

			0	1	2	5	6	7
P_1	(0, 1)	*a′b′*	×	×				
P_2	(0, 2)	*a′c′*	×		×			
P_3	(1, 5)	*b′c*		×		×		
P_4	(2 6)	*bc′*			×		×	
P_5	(5, 7)	*ac*				×		×
P_6	(6, 7)	*ab*					×	×

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	X	X				
P_2	(0, 2)	$a'c'$	X		X			
P_3	(1, 5)	$b'c$		X		X		
P_4	(2 6)	bc'			X		X	
P_5	(5, 7)	ac				X		X
P_6	(6, 7)	ab					X	X

- The next step is to reduce *P* to a minimum sum of products. This is easy because there are no complements.

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

$$\begin{aligned} P &= (P_1 + P_2P_3)(P_4 + P_2P_6)(P_5 + P_3P_6) \\ &= (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6) \\ &= P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6 \\ &\quad + P_1P_2P_3P_6 + P_2P_3P_4P_6 + P_2P_3P_6 \end{aligned}$$

- Next, we use $X + XY = X$

$$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6 + P_1P_2P_3P_6 + P_2P_3P_4P_6 + P_2P_3P_6$$

$$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$$

- find those terms which contain a minimum number of variables.
- Each of these terms represents a solution with a minimum number of prime implicants.

$$P = P_1P_4P_5 + P_2P_3P_6$$

- Steps:
- **1.** Reduce the prime implicant chart by eliminating the essential prime implicant rows and the corresponding columns.
- **2.** Label the rows of the reduced prime implicant chart *P*1, *P*2, *P*3, etc.
- **3.** Form a logic function *P* which is true when all columns are covered. *P* consists of a product of sum terms, each sum term having the form (*Pi*0 + *Pi*1 + · · ·), where *Pi*0, *Pi*1 . . . represent the rows which cover column *i*.
- **4.** Reduce *P* to a minimum sum of products by multiplying out and applying $X + XY = X$.

- **5.** Each term in the result represents a solution, that is, a set of rows which covers all of the minterms in the table. To determine the minimum solutions (as defined In lect 06), find those terms which contain a minimum number of variables. Each of these terms represents a solution with a minimum number of prime implicants.
- **6.** For each of the terms found in step 5, count the number of literals in each prime implicant and find the total number of literals. Choose the term or terms which correspond to the minimum total number of literals, and write out the corresponding sums of prime implicants.