Lecture (05)

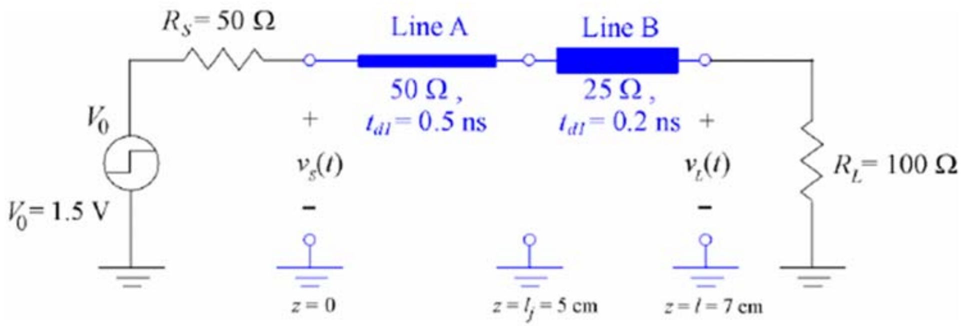
Steady-state Response of Transmission Lines (Cont,..)

By:

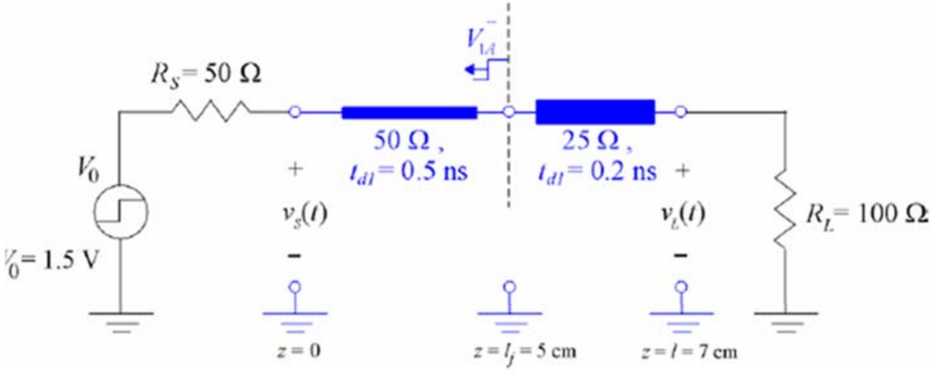
Dr. Ahmed ElShafee

Example 03 (cascaded TXs)

Consider a system shown in Fig. Find the terminal voltages $v_S(t)$, $v_L(t)$.



(1) At $z=l_j,\ t=t_{d1}$:



$$v_{1A}^+(l_j,t_{d1})+v_{1A}^-(l_j,t_{d1})=v_{1B}^+(l_j,t_{d1})$$

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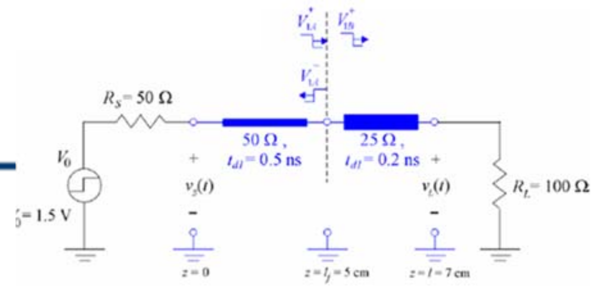
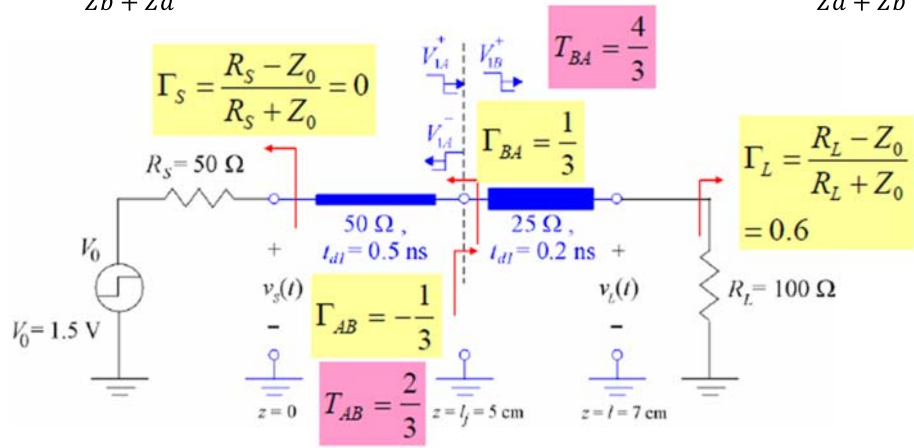


$$v_{1A}^+(l_j,t_{d1})+v_{1A}^-(l_j,t_{d1})=v_{1B}^+(l_j,t_{d1})$$

$$\Rightarrow 1+\frac{v_{1A}^-(l_j,t_{d1})}{v_{1A}^+(l_j,t_{d1})}=\frac{v_{1B}^+(l_j,t_{d1})}{v_{1A}^+(l_j,t_{d1})},$$

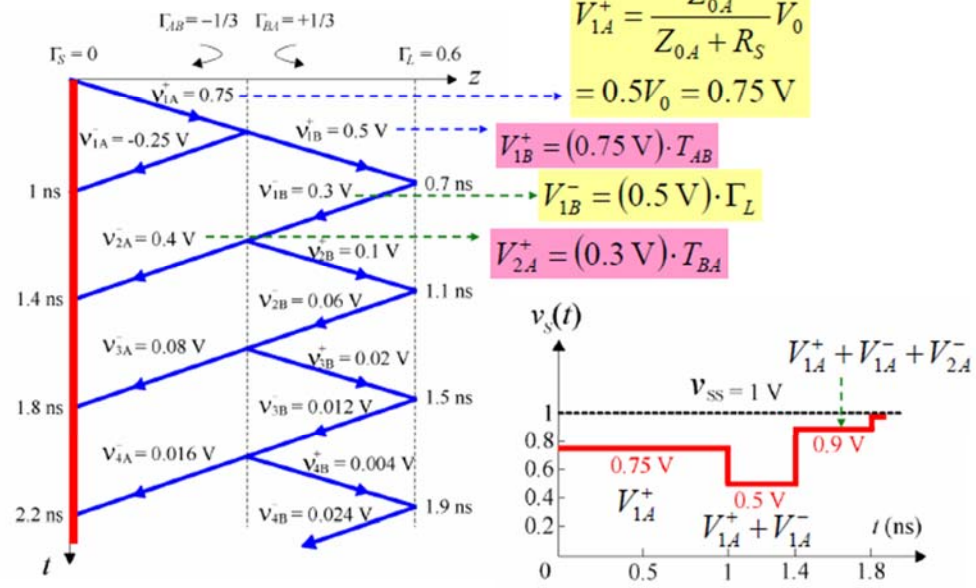
$$\Rightarrow 1+\Gamma_{AB}=T_{AB}$$ …Transmission coefficient

$$\Gamma AB = \frac{Zb - Za}{Zb + Za} \qquad \Gamma BA = \frac{Za - Zb}{Za + Zb}$$



$\Gamma_L = 0.6$ $\Gamma_S = 0$ $T_{BA} = \frac{4}{3}$ $Rs = 50$ $Rl = 100$

$\Gamma_{BA} = \frac{1}{3}$ $\Gamma_{AB} = -\frac{1}{3}$ $T_{AB} = \frac{2}{3}$ $Za = 50$ $Zb = 25$



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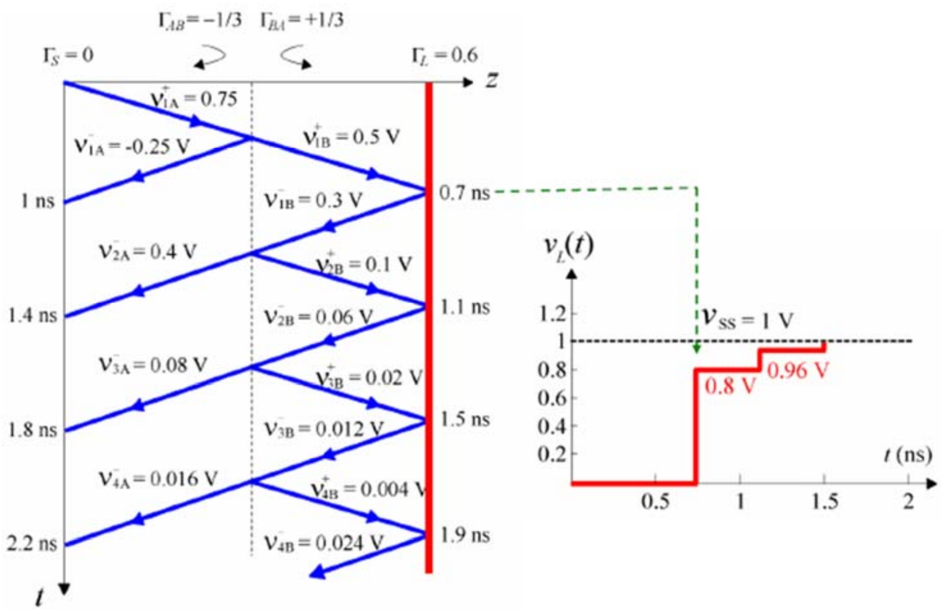
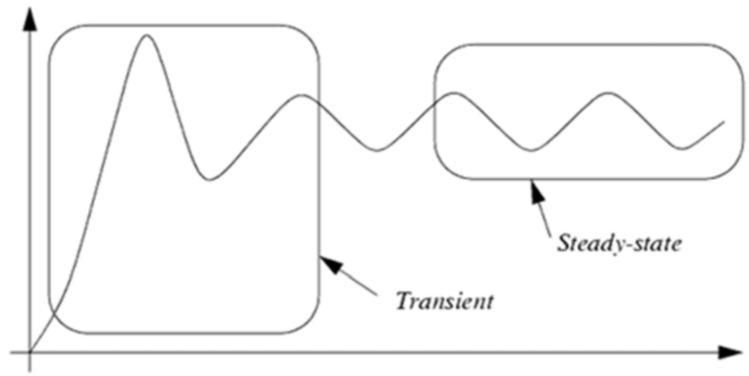


Transient response

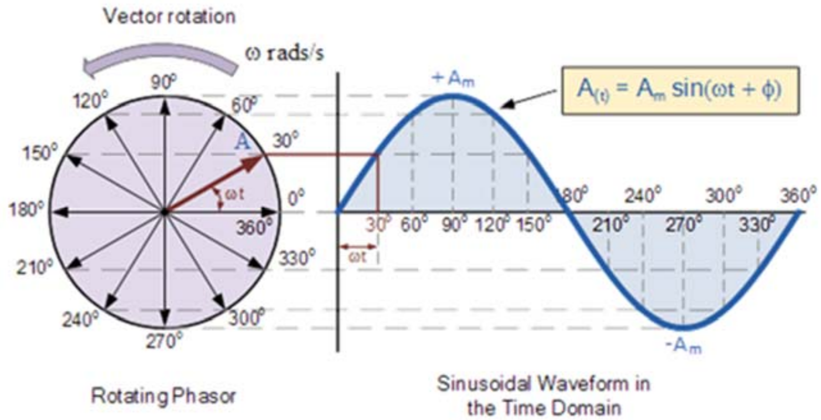
- Reasons to consider the response of TX lines to sinusoidal excitations.

  Power & communication signals are transmitted as (modified) sinusoids.
- Natural (transient) response will decay rapidly.
- Forced (steady-state) response is supported by the source, and will continue indefinitely



Note: the transient response is predicted with the homogeneous solution. The steady state response in mainly predicted with the particular solution, although in some cases the homogeneous solution might have steady state effects, such as a non-decaying oscillation.

- By using phasors, we can introduce complex reflection coefficient and line impedance to simplify the analysis.



Using Euler's identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$\cos\theta = \mathrm{Re}\{e^{j\theta}\} \qquad \sin\theta = \mathrm{Im}\{e^{j\theta}\}$$

Real part — Imaginary part

The input sinusoid *v(t)* can be rewritten as:

$$v(t) = V_m \cos(\omega t + \phi) = V_m \,\mathrm{Re}\{e^{j(\omega t + \phi)}\}$$

$$= V_m \,\mathrm{Re}\{e^{j\omega t} e^{j\phi}\} = \mathrm{Re}\{V_m e^{j\phi} e^{j\omega t}\}$$

Contains magnitude and phase info

The **phasor for *v(t)*** is defined as

$$V = V_m e^{j\phi} = \mathcal{P}[v(t)]$$

$\mathcal{P}$: phase operator or phasor transform

proof 01: find wave equation propagation through TX as TDE in z domain

- When the steady-state due to a sinusoidal excitation is reached, the voltage $v(z,t)$ and the current $i(z,t)$ on the transmission line must also be sinusoidal waves, which can be represented by the z-dependent phasors $V(z)$, $I(z)$:

$$v(z,t) = \mathrm{Re}\{V(z)\cdot e^{j\omega t}\}, \quad i(z,t) = \mathrm{Re}\{I(z)\cdot e^{j\omega t}\}$$

- Phasor: complex function of z
- But we know that

$$\frac{\partial}{\partial z} v(z,t) = -L \frac{\partial}{\partial t} i(z,t) \qquad \text{…1st-order PDE}$$

- $$\Rightarrow \frac{\partial}{\partial z}\mathrm{Re}\{V(z)\cdot e^{j\omega t}\} = -L\frac{\partial}{\partial t}\mathrm{Re}\{I(z)\cdot e^{j\omega t}\}$$

$$\Rightarrow \mathrm{Re}\left\{\left[\frac{d}{dz}V(z)\right]e^{j\omega t}\right\} = -L\cdot\mathrm{Re}\left\{I(z)\left[\frac{\partial}{\partial t}e^{j\omega t}\right]\right\}$$

$$\Rightarrow \mathrm{Re}\left\{\left[\frac{d}{dz}V(z)\right]e^{j\omega t}\right\} := -L\cdot\mathrm{Re}\{I(z)\cdot j\omega e^{j\omega t}\}$$

$$\boxed{\frac{d}{dz}V(z) = -j\omega L\cdot I(z)}$$ …1st-order ODE

- from

$$\boxed{\frac{d}{dz}V(z) = -j\omega L\cdot I(z)}$$ …1st-order ODE

- We can conclude that

PDE's		ODE's
$\frac{\partial}{\partial z}v(z,t) = -L\frac{\partial}{\partial t}i(z,t)$	→	$\frac{d}{dz}V(z) = -j\omega L\cdot I(z)$
$\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t)$	→	$\frac{d}{dz}I(z) = -j\omega C\cdot V(z)$

- Find 2[nd] derivative $\frac{d}{dz}V(z) = -j\omega L\cdot I(z)$
- $\frac{d^2V(z)}{dz^2} = -j\omega L\,\frac{d\,I(z)}{dz}$
- But $\frac{d}{dz}I(z) = -j\omega C\cdot V(z)$
- Substitute
- $\frac{d^2V(z)}{dz^2} = -\omega^2 LC\,V(z)$
- Set $\beta = \omega\sqrt{LC}$
- $\boxed{\frac{d^2}{dz^2}V(z) = -\beta^2 V(z)}$

PDE's		ODE's
$\frac{\partial}{\partial z}v(z,t) = -L\frac{\partial}{\partial t}i(z,t)$	→	$\frac{d}{dz}V(z) = -j\omega L\cdot I(z)$
$\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t)$	→	$\frac{d}{dz}I(z) = -j\omega C\cdot V(z)$
$\frac{\partial^2}{\partial z^2}v(z,t) = LC\frac{\partial^2}{\partial t^2}v(z,t)$	→	$\boxed{\frac{d^2}{dz^2}V(z) = -\beta^2 V(z)}$ ↓ $\beta = \omega\sqrt{LC}$
$\frac{\partial^2}{\partial z^2}i(z,t) = LC\frac{\partial^2}{\partial t^2}i(z,t)$	→	$\frac{d^2}{dz^2}I(z) = -\beta^2 I(z)$

Proof 02 : TX waves equations in phasor form, and calculating Z_0

- $$\beta = \omega\sqrt{LC}$$
  $$vp = \frac{1}{\sqrt{LC}}$$
  $$\beta = \omega/vp$$
  $$vp = \frac{z}{t}$$
  $$\beta = \omega t/z$$
  $$\beta z = \omega t$$

- $$v(z,t) = Re\{v(z)e^{j\omega t}\}$$
  $$v(z) = vm\ e^{j\beta z}$$
  $$v(z) = v^+\ e^{-j\beta z}e^{j\emptyset^+} + v^-\ e^{j\beta z}e^{j\emptyset^-}$$
  $$v(z,t) = \mathrm{Re}\left\{|V^+|e^{j\phi^+}e^{-j\beta z}e^{j\omega t} + |V^-|e^{j\phi^-}e^{j\beta z}e^{j\omega t}\right\}$$
  $$v(z,t) = |V^+|\cos(\omega t - \beta z + \phi^+) + |V^-|\cos(\omega t + \beta z + \phi^-)$$

- $$v(z,t) = |V^+|\cos(\omega t - \beta z + \phi^+) + |V^-|\cos(\omega t + \beta z + \phi^-)$$

$$v(z,t) = |V^+|\cos\left[\omega\left(t - \frac{z}{\omega/\beta}\right) + \phi^+\right] + |V^-|\cos\left[\omega\left(t + \frac{z}{\omega/\beta}\right) + \phi^-\right]$$

$$f^+(t - z/v_p) \qquad f^-(t + z/v_p)$$

$\beta = \omega/vp$

$$\beta z = \omega t$$
$$\frac{\omega}{\beta} = \frac{z}{t} = vp$$

- To solve the current phasor *I (z)*
  $$V(z) = V^+(z) + V^-(z) = V^+e^{-j\beta z} + V^-e^{j\beta z}$$
- From previous proof
  $$\frac{d}{dz}V(z) = -j\omega L \cdot I(z)$$
- Then
  $$V(z) = \Rightarrow -j\beta V^+e^{-j\beta z} + j\beta V^-e^{j\beta z} = -j\omega L \cdot I(z)$$

$\beta z = \omega t$

$$V(z) = -j\beta V^{+}e^{-j\beta z} + j\beta V^{-}e^{j\beta z} = -j\omega L \cdot I(z)$$

- Isolate i(z)

$$I(z) = \frac{\beta}{\omega L}V^{+}e^{-j\beta z} - \frac{\beta}{\omega L}V^{-}e^{j\beta z}$$

- But $\beta = \omega\sqrt{LC}$, $\frac{\beta}{\omega L} = \sqrt{C/L} = \frac{1}{z_0}$ and $V(z) = V^{+}(z) + V^{-}(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}$

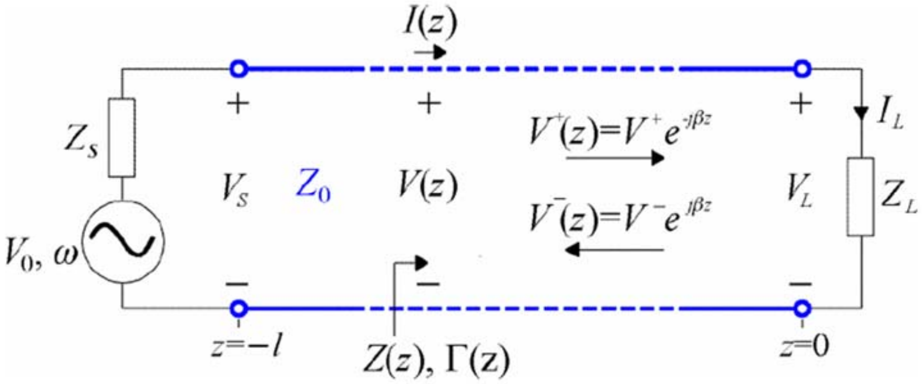
$$I(z) = I^{+}(z) + I^{-}(z) = \frac{V^{+}(z)}{Z_0} - \frac{V^{-}(z)}{Z_0}$$

$$Z_0 = \frac{V^{+}(z)}{I^{+}(z)} = -\frac{V^{-}(z)}{I^{-}(z)} \longleftrightarrow Z_0 = \frac{v^{+}(z,t)}{i^{+}(z,t)} = -\frac{v^{-}(z,t)}{i^{-}(z,t)}$$

V-I ratio of a single wave.

Reflection at Discontinuity

- Consider a lossless transmission line of characteristic impedance $Z_0 \in R$, propagation constant β , driven by a sinusoidal source of angular frequency ω , and terminated by an impedance $Z_L \in C$



Proof 01: Both Γ(*z*) and *Z*(*z*) are complex periodic functions of period λ/2 .

- We know that

$$Z_0 = \frac{V^{+}(z)}{I^{+}(z)} = -\frac{V^{-}(z)}{I^{-}(z)}$$

- the boundary condition requires

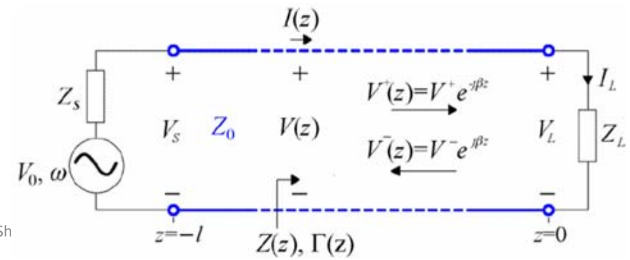
$$\left.\frac{V^{+}(z)+V^{-}(z)}{I^{+}(z)+I^{-}(z)}\right|_{z=0} = Z_L$$

- Therefore, a reflected wave $V^{-}(z)$ must be generated if the load is not matched to the line($Z_0 \neq Z_L$)

- the voltage reflection coefficient Γ(*z*) and the line impedance *Z*(*z*) (looking toward the load at *z* = 0) depend on the point of observation *z*

$$\Gamma(z) \equiv \frac{V^{-}(z)}{V^{+}(z)} = \frac{V^{-}e^{j\beta z}}{V^{+}e^{-j\beta z}} = \frac{V^{-}}{V^{+}}e^{j2\beta z}, \Rightarrow$$

$$\Gamma(z) = \Gamma_L e^{j2\beta z}, \qquad \Gamma_L = \Gamma(0) = \frac{V^{-}}{V^{+}}$$



$$Z(z) \equiv \frac{V(z)}{I(z)} = \frac{V^+(z)+V^-(z)}{I^+(z)+I^-(z)} = \frac{V^+e^{-j\beta z}+V^-e^{j\beta z}}{(V^+/Z_0)e^{-j\beta z}+(-V^-/Z_0)e^{j\beta z}}, \Rightarrow$$

$$Z(z) = Z_0\frac{V^+e^{-j\beta z}+V^-e^{j\beta z}}{V^+e^{-j\beta z}-V^-e^{j\beta z}}, \quad Z_L = Z(0) = Z_0\frac{V^++V^-}{V^+-V^-}$$

- Instead of using the voltage amplitudes $V+$ and $V-$, it is more convenient to express Γ(*z*), *Z*(*z*) by the characteristic and load impedances Z_0 and Z_L.
- Divide right by V^+ $\quad Z_L = Z_0\frac{V^++V^-}{V^+-V^-} = Z0\frac{1+\frac{V-}{V+}}{1-\frac{V-}{V+}}$
- sub $\quad \Gamma_L = \Gamma(0) = \frac{V^-}{V^+}$
- then $\quad Z_L = Z_0\frac{1+\Gamma_L}{1-\Gamma_L}$
- reformat $\quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ Look at next slide

$$Z_L = Z_0\frac{1+\Gamma_L}{1-\Gamma_L}$$

$$Z_L = Z_0\frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\frac{Z_L}{Z_0} = \frac{1+\Gamma_L}{1-\Gamma_L}$$

$$(1-\Gamma)\frac{Z_L}{Z_0} = 1+\Gamma_L$$

$$\frac{Z_L}{Z_0} - \frac{Z_L}{Z_0}\Gamma_L = 1+\Gamma_L$$

$$\frac{Z_L}{Z_0} - 1 = \frac{Z_L}{Z_0}\Gamma_L + \Gamma$$

$$\frac{Z_L}{Z_0} - 1 = \Gamma_L\left(\frac{Z_L}{Z_0}+1\right)$$

$$\Gamma_L = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} \quad X\ Z_0$$

$$= \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- But $\quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$ $\quad$ and $\quad \Gamma(z) = \Gamma_L e^{j2\beta z}$,
- Sub on previous $\quad \Gamma(z) = \frac{Z_L - Z_0}{Z_L + Z_0}e^{j2\beta z}$
- Both Γ(*z*) is a complex periodic functions of period λ/2

$$\Gamma(z) = \frac{Z_L - Z_0}{Z_L + Z_0}e^{j2\beta z}$$

- We find before that

$$Z(z) = Z_0\frac{V^+e^{-j\beta z}+V^-e^{j\beta z}}{V^+e^{-j\beta z}-V^-e^{j\beta z}}$$

- By dividing V^+ for the numerator and the denominator and sub $\quad \Gamma_L = \frac{V^-}{V^+}$

$$Z(z) = Z_0\frac{e^{-j\beta z}+\Gamma_L e^{j\beta z}}{e^{-j\beta z}-\Gamma_L e^{j\beta z}}$$

$$= Z_0\frac{(\cos\beta z - j\sin\beta z)+\Gamma_L(\cos\beta z + j\sin\beta z)}{(\cos\beta z - j\sin\beta z)-\Gamma_L(\cos\beta z + j\sin\beta z)}$$

- By dividing cosβ*z* for the numerator and the denominator

$$Z(z) = Z_0 \frac{(1 - j\tan\beta z) + \Gamma_L (1 + j\tan\beta z)}{(1 - j\tan\beta z) - \Gamma_L (1 + j\tan\beta z)}$$

$$Z_0 \frac{(1+\Gamma_L) - j(1-\Gamma_L)\tan\beta z}{(1-\Gamma_L) - j(1+\Gamma_L)\tan\beta z}$$

- Divide by 1-Γ_L ,& sub from $Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}, \Rightarrow$
- Then mult numerator and the denominator by Z_0

$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan(\beta z)}{Z_0 - jZ_L \tan(\beta z)}$$

- *Z*(*z*) is a complex periodic functions of period λ/2 .