Lecture (02)

Transmission Lines Fundamentals

By:

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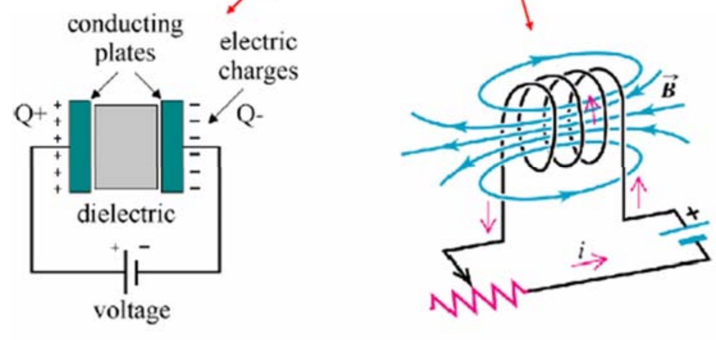
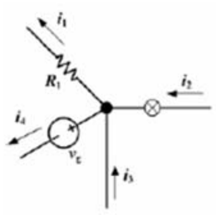
- Models of linear circuit elements:

$v = R \cdot i$ $\quad$ $i = C \cdot \frac{d}{dt} v$ $\quad$ $v = L \cdot \frac{d}{dt} i$

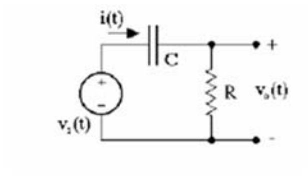


Kirchhoff's laws :

$$\sum_k i_k = 0$$

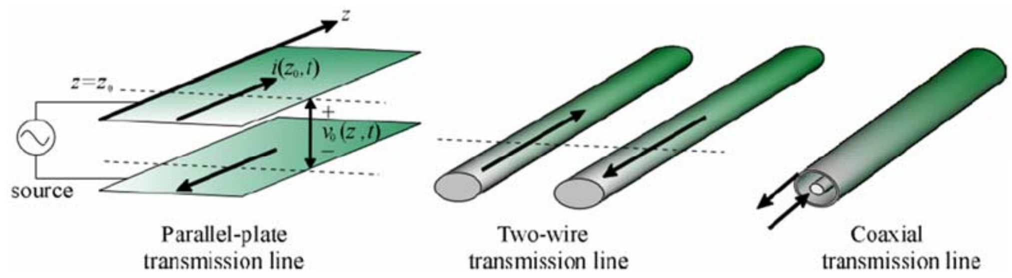


$$\sum_k v_k = 0$$



Equivalent Circuit and Equations of Transmission Lines

- Geometry of transmission lines



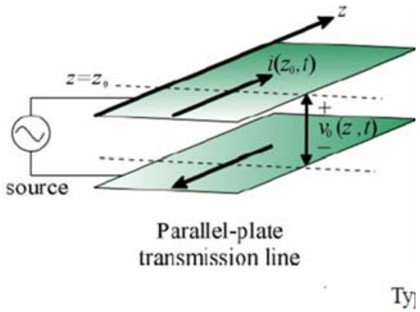
Typical types of transmission lines.

- Typical transmission lines consist of two long conductors separated by some insulating material

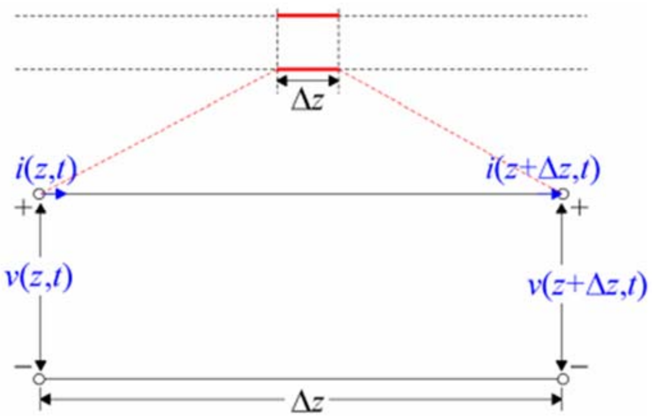
- when the two electrodes of a (voltage or current) source are connected to the transmission line.

At any transverse plane

- $z = z0$
- voltage drop $v(z_0,t)$ between the two conductors exists
- currents ± $i(z_0,t)$ with equal magnitude but opposite directions flow along the two conductors,



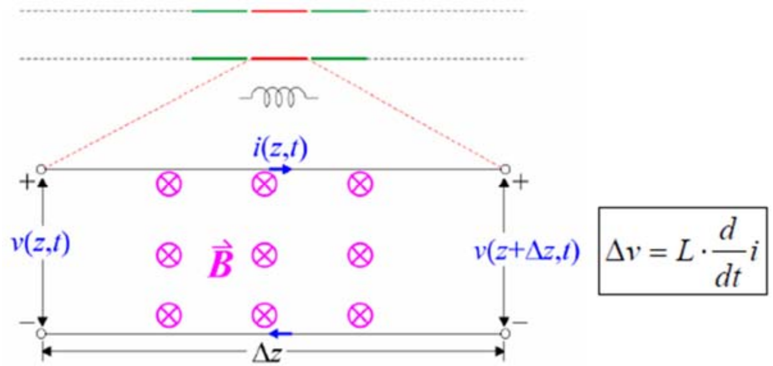
- Since the voltage, current can vary with z, ⇒ use of distributed circuit model, i.e. a TX line consists of tiny many coupled lines of tiny length Δz



Equivalent circuit

- Since the voltage and current of a transmission line vary with position z (and time t), we have to characterize it by a "distributed" circuit model.
- (1) Consider an tiny line of length Δz, the currents set up magnetic field between the conductors (by Ampere's law), causing magnetic flux.
- (2)When currents are time-varying, so is the magnetic flux, and a voltage variation "along" the conductor (electromotive force) is induced (by Faraday's law), in an attempt to drive the currents oppositely (by Lenz's law).

- Currents on a short line set up magnetic field (Ampere's law), ⇒ magnetic flux
- Time-varying current (flux), ⇒ voltage changes along the line (Faraday's law) to counter the change of current (Lenz's law), ⇒ series inductor

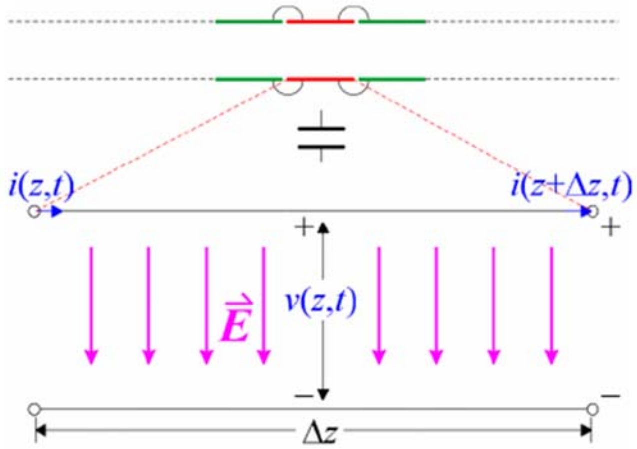


- (3)This behavior can be modeled by a **series inductor**

$$\left(v = L \cdot \frac{d}{dt} i\right).$$

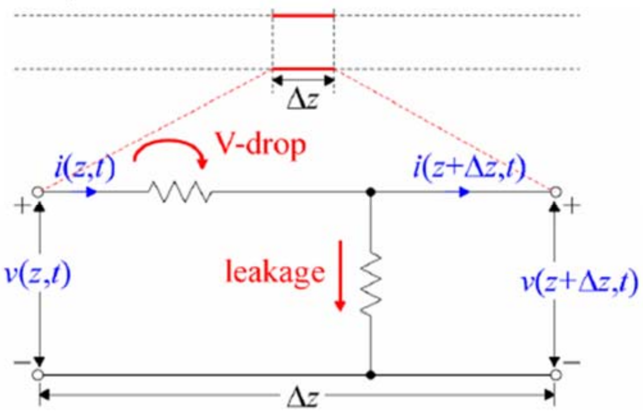
- **(4)** Meanwhile, two separated conductors form a capacitor. Since the upper and lower conductors of adjacent tiny lines are connected respectively, the capacitive behavior of an tiny line can be modeled by a **shunt capacitor**.

- The upper and lower conductors of the adjacent short lines are connected respectively, ⇒ shunt capacitor

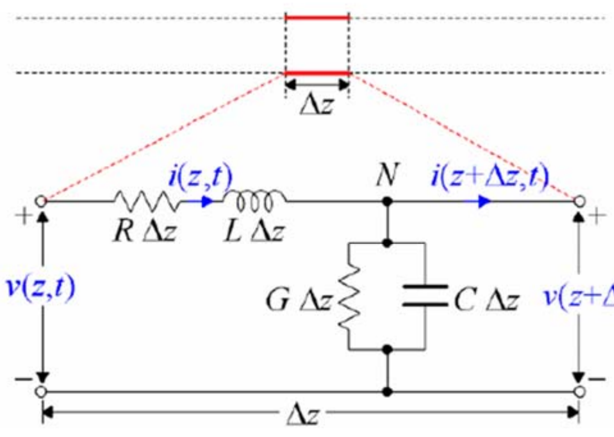


$$\Delta i = C \cdot \frac{d}{dt} v$$

- Imperfect conducting materials, ⇒ voltage drop along the conducting line, ⇒ series resistor
- Imperfect insulating materials, ⇒ leakage current between conductors, ⇒ shunt conductor

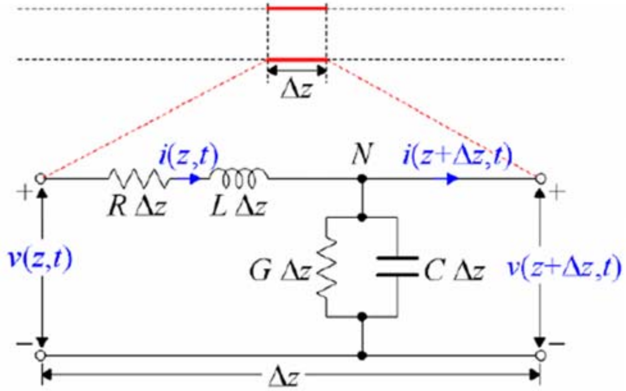


- (**finally**) which can be modeled by a series resistor and a shunt conductor, respectively



Equivalent circuit of a real transmission line.

- where R, L, G, C represent resistance, inductance, conductance, and capacitance per unit length.
- Transmission line is lossless in the absence of R and G.



Equivalent circuit of a real transmission line.

- **<Comment>**

1) By using the equivalent circuit, analysis of electric and magnetic vector fields is substituted by that of scalar voltage between and current along the line, greatly simplifying the math.

2) Values of R, L, G, C depend on geometry and material characteristics of transmission line. We will discuss how to calculate them in the subsequent lectures.

Total and partial derivative

- $\partial f/\partial x$ is the partial derivative: f is differentiated w.r.t. to x while *all* other variables are considered constants in x.
- df/dx the is total derivative: f is differentiated w.r.t. to x while *nothing* is assumed about the other variables; they are considered variables in x. (some variables might be, in fact, constants in x.)

$$f(x,v) = x^2 + v(x)$$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial v} = 1$$

$$\frac{df}{dx} = 2x + \frac{\partial v(x)}{\partial x}$$

Limits and differentiation

- The geometric meaning of the derivative

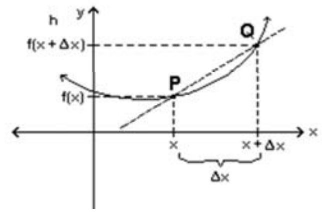
$$f'(x) = \frac{df(x)}{dx}$$

is the slope of the line tangent to $y = f(x)$ at x.

Let's look for this slope at P:

The **secant** line through P and Q has slope

$$\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$



We can approximate the **tangent** line through P by moving Q towards P, decreasing Δx. In the limit as $\Delta x \to 0$, we get the tangent line through P with slope

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

We define

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}^{*}.$$

* If the limit as $\Delta x \to 0$ at a particular point does not exist, $f'(x)$ is undefined at that point.

We derive all the basic differentiation formulas using this definition.

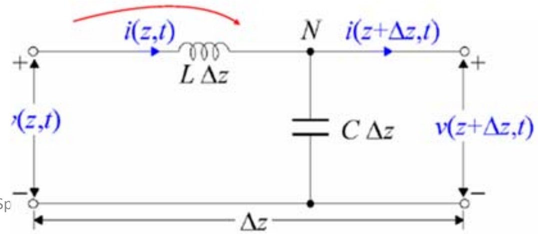
Proof 01: 1.find Lossless transmission line waves propagation equations
2. finding wave propagation speed

- Assuming $R = 0$, $G = 0$ (lossless line) in
- Applying ohm's law

$$v_L(t) = (L\Delta z)\cdot \frac{d}{dt} i_L(t)$$

- Applying Kirchhoff's voltage law:

$$v(z+\Delta z,t) = v(z,t) - (L\Delta z)\frac{\partial}{\partial t} i(z,t)$$



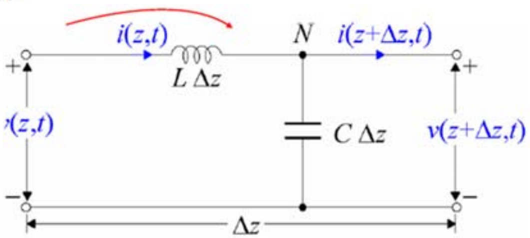
$$v(z+\Delta z,t) = v(z,t) - (L\Delta z)\frac{\partial}{\partial t} i(z,t)$$

$$\Rightarrow v(z+\Delta z,t) - v(z,t) = -(L\Delta z)\frac{\partial}{\partial t} i(z,t)$$

$$\Rightarrow \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -(L)\frac{\partial}{\partial t} i(z,t)$$

Let $\Delta z \to 0$,

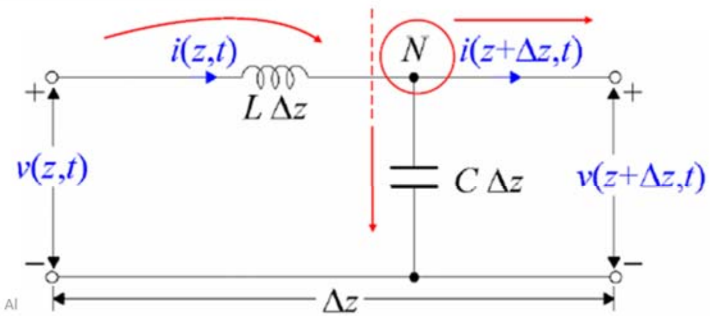
$$\Rightarrow \boxed{\frac{\partial}{\partial z} v(z,t) = -L\frac{\partial}{\partial t} i(z,t)}$$



1st-order PDE with 2 unknown functions v and i

- Applying ohm's law $i_C(t) = (C\Delta z)\cdot \frac{d}{dt} v_C(t)$
- Applying Kirchhoff's current law @ N:

$$\boxed{i(z,t) = i(z+\Delta z,t) + (C\Delta z)\frac{\partial}{\partial t} v(z+\Delta z,t)}$$



$$i(z,t) = i(z+\Delta z,t) + (C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

$$\Rightarrow i(z,t) - i(z+\Delta z,t) = (C\Delta z)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

$$\Rightarrow \frac{i(z,t)-i(z+\Delta z,t)}{\Delta z} = (C)\frac{\partial}{\partial t}v(z+\Delta z,t)$$

Let $\Delta z \to 0$,

$$\Rightarrow \boxed{\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t)}$$

1st-order PDE with 2 unknown functions v and i

- Taking $\frac{\partial}{\partial z}$ for both sides of $\frac{\partial}{\partial z}v(z,t) = -L\frac{\partial}{\partial t}i(z,t)$

$$\frac{\partial}{\partial z}v(z,t) = -L\frac{\partial}{\partial t}i(z,t), \Rightarrow \frac{\partial^2}{\partial z^2}v(z,t) = -L\frac{\partial^2}{\partial z \partial t}i(z,t)$$

Taking $\frac{\partial}{\partial t}$ for both sides of $\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t)$

$$\frac{\partial}{\partial z}i(z,t) = -C\frac{\partial}{\partial t}v(z,t), \Rightarrow \frac{\partial^2}{\partial z \partial t}i(z,t) = -C\frac{\partial^2}{\partial t^2}v(z,t)$$

$$\frac{\partial^2}{\partial z^2}v(z,t) = -L\frac{\partial^2}{\partial z \partial t}i(z,t)$$

$$\frac{\partial^2}{\partial z \partial t}i(z,t) = -C\frac{\partial^2}{\partial t^2}v(z,t)$$

- Combine the two equations

$$\Rightarrow \frac{\partial^2}{\partial z^2}v(z,t) = LC\frac{\partial^2}{\partial t^2}v(z,t)$$

- 2nd-order PDE with 1 unknown function v

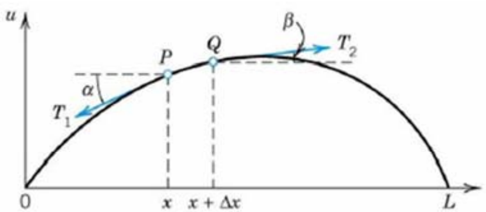
$$\boxed{\frac{\partial^2}{\partial z^2}v(z,t) = LC\frac{\partial^2}{\partial t^2}v(z,t)}$$

Similarly, $i(z,t)$ is governed by the same PDE:

$$\boxed{\frac{\partial^2}{\partial z^2}i(z,t) = LC\frac{\partial^2}{\partial t^2}i(z,t)}$$

- Compare $\frac{\partial^2}{\partial z^2}v(z,t) = LC\frac{\partial^2}{\partial t^2}v(z,t)$ (TX line eq.)

  with $\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t)$ (1-D wave eq.)



$v(z,t)$ ~ a wave propagating with velocity $\boxed{v_p \equiv \frac{1}{\sqrt{LC}}}$

Proof 02 : To verify the fact

$$vp = 1/\sqrt{LC}$$

any function $f(\cdot)$ of variable $\tau = t - \frac{z}{v_p}$ is a

let $v(z,t) = f(\tau), \Rightarrow$

$$\frac{\partial v}{\partial z} = \frac{df}{d\tau}\cdot\frac{\partial \tau}{\partial z} = -\frac{1}{v_n}\frac{df}{d\tau} \qquad \frac{\partial \tau}{\partial z} = -\frac{1}{v_p}$$

$$\frac{\partial^2 v}{\partial z^2} = -\frac{1}{v_p}\frac{\partial}{\partial z}\left(\frac{df}{d\tau}\right) =$$

$$-\frac{1}{v_p}\frac{\partial}{\partial z}\frac{df}{d\tau} = -\frac{1}{v_p}\frac{\partial \tau}{\partial \tau}\frac{\partial}{\partial z}\left(\frac{df}{d\tau}\right) =$$

$$-\frac{1}{v_p}\frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau}\left(\frac{df}{d\tau}\right) = \frac{1}{v_p{}^2}\frac{\partial}{\partial \tau}\left(\frac{df}{d\tau}\right) = \frac{1}{v_p{}^2}\frac{\partial^2 f}{\partial^2 \tau}$$

$$= \frac{1}{v_p^2} f''(\tau)\Big|_{\tau = t - z/v_p}$$

$$\frac{\partial^2 v}{\partial^2 z} = \frac{1}{v_p{}^2}\frac{\partial^2 f}{\partial^2 \tau} = \frac{1}{v_p^2} f''(\tau)\Big|_{\tau=t-z/v_p}$$

let $v(z,t) = f(\tau), \Rightarrow$

$$\frac{\partial v}{\partial t} = \frac{df}{d\tau}\cdot\frac{\partial \tau}{\partial t} = \frac{df}{d\tau} \qquad \because \frac{\partial \tau}{\partial t} = 1,$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{df}{d\tau}\right) = \frac{d^2 f}{d\tau^2} = f''(\tau)\Big|_{\tau=t-z/v_p}$$

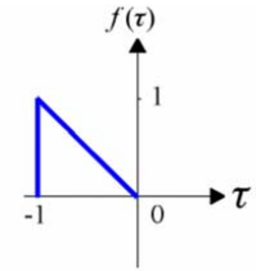
$$\frac{\partial^2 v}{\partial z^2} = \frac{f''(\tau)}{v_p^2} = \frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2}$$

consistent with $\boxed{\frac{\partial^2}{\partial z^2}v(z,t) = LC\frac{\partial^2}{\partial t^2}v(z,t)}$ regardless of the functional form $f(\cdot)$

Proof 03: wave propagation through Tx doesn't change its shape

- the waveform through tx line is displaced by constant value within t_0 sec (speed equals = v m/s) without changing its shape
- *f (t – z /Vp)* represents a distortion-free wave traveling in the +z direction with phase velocity *Vp*

$$f(t - z/v_p) = f(\tau) = \begin{cases} -\tau, \text{if } -1 < \tau < 0 \\ 0, \text{otherwise} \end{cases}$$



$$f(t - z/v_p) = f(\tau) = \begin{cases} -\tau, \text{if } -1 < \tau < 0 \\ 0, \text{otherwise} \end{cases},$$

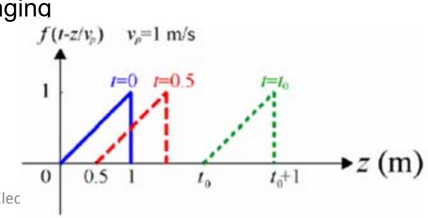
- @ $v_p = 1$ m/s. At $t = 0$

$$f(t - z/v_p) = f(-z) = \begin{cases} z, \text{if } 0 < z < 1 \\ 0, \text{otherwise} \end{cases}$$

@ $v_p = 1$ m/s. At $t = 0.5$ sec,

$$f(t - z/v_p) = f(0.5 - z) = \begin{cases} z - 0.5, \text{if } 0.5 < z < 1.5 \\ 0, \text{otherwise} \end{cases}$$

the waveform is displaced by +0.5 m within 0.5 sec (speed equals = *Vp*= 1 m/s) without changing its shape



$$f\left(t - z/v_p\right) = f(\tau) = \begin{cases} -\tau, \text{if } -1 < \tau < 0 \\ 0, \text{otherwise} \end{cases},$$

- The wave propagation property continues for any $t = t0$ sec, where

$$f\left(t - z/v_p\right) = f\left(t_0 - z\right) = \begin{cases} z - t_0, \text{if } t_0 < z < 1 + t_0 \\ 0, \text{otherwise} \end{cases}.$$