Lecture (05)

Step Response of a RLC

By:

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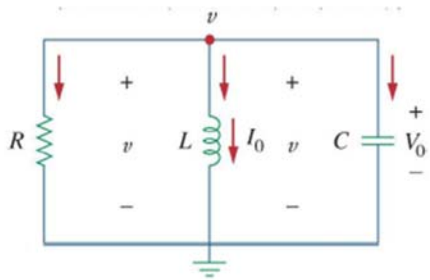
Source Free RLC parallel circuit

$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt$

$v(0) = V_0$



- Applying KCL @ V

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0$$

- Divide y C and Take the derivative to get

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

- Need two initial conditions $v(0)$ and $\frac{dv(0)}{dt}$
  Assume $v = A\,e^{st}$ to get
- $V(0) = A$
- $\frac{dv}{dt} = A\,s\,est$
- $\frac{dv(0)}{dt} = A\,s$
- $\frac{d^2v}{dt^2} = A\,s^2\,est$
- $\frac{d^2v(0)}{dt^2} = As^2$
- $A\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = 0$
- $\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = 0$

- General solution for 2nd order polynomial equation

$$ax^2 + bx + c = 0$$

$$(x - x_1)(x - x_2) = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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- $\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = 0$
- The roots will give us the natural frequencies $s1,2$:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \text{ and } s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

- General solution will be

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Case 1: Overdamped $(\alpha > \omega_0)$ $L > 4R^2C.$

$s_{1,2}$ are real: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o}$$

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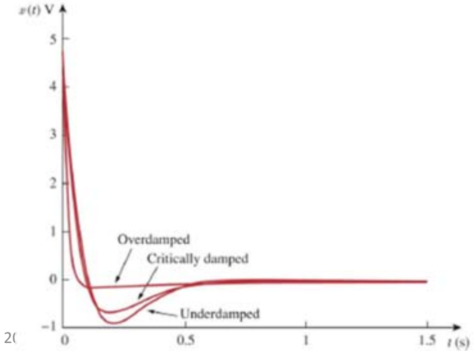
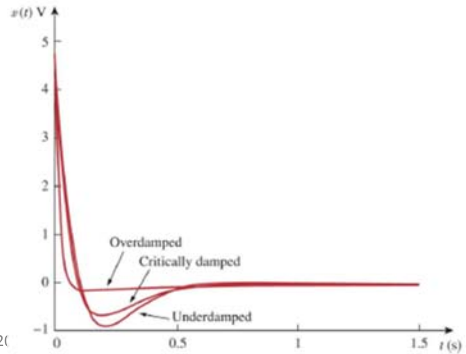


Case 2: Critically damped $(\alpha = \omega_0)$ $L = 4R^2C.$

$s_1 = s_2 = -\alpha$:

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$



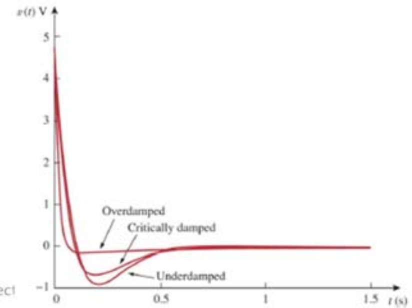
Case 3: Underdamped $(\alpha < \omega_0)$ $L < 4R^2C.$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$s_{1,2} = -\alpha \pm j\omega_d$ where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$:

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

ω_d: damping frequency



- $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

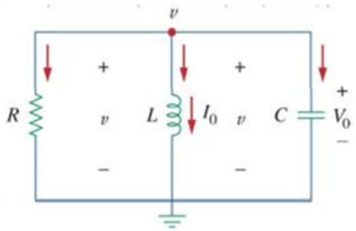
The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from

$$v(0) = V_0$$

We find the second term by

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$



Summary

$$\alpha = \frac{1}{2RC} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

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$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o}$ $\qquad$ $s_{1,2} = -\alpha \pm j\omega_d$ where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$s_1 = s_2 = -\alpha$

- $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ (Overdamped) $\alpha > \omega_0$
- $v(t) = (A_1 t + A_2) e^{-\alpha t}$ (Critically damped) $\alpha = \omega_0$
- $v(t) = (A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) e^{-\alpha t}$ (Underdamped) $\alpha < \omega_0$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

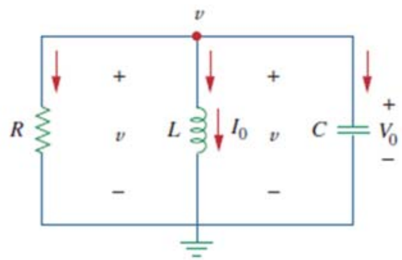
Init. Cond.

$$v(0) = V_0$$

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

Example 01

In the parallel circuit of Fig. , find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.



$v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF.
$R = 1.923\ \Omega$,

- Case 1

If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

We now apply the initial conditions to get A_1 and A_2.

$$v(0) = 5 = A_1 + A_2$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating $v(t) = A_1 e^{-2t} + A_2 e^{-50t}$

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2$$

- $A_1 = -0.2083$

$A_2 = 5.208.$

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t}$$

$v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF.
$R = 1.923\ \Omega$,

- Case 2: $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped.

$s_1 = s_2 = -10$,

$$v(t) = (A_1 + A_2 t)e^{-10t}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. $v(t) = (A_1 + A_2 t)e^{-10t}$

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2$$

$A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t} \text{ V}$$

- Case 03 :

When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

But differentiating $v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$

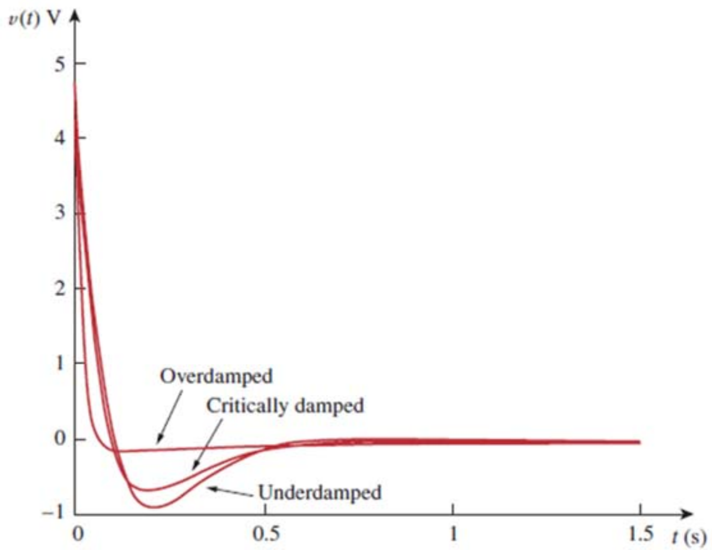
$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2$$

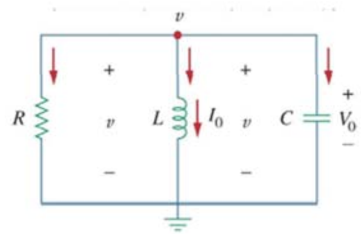
$A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t}$$



Example 02

In Fig. , let $R = 2\ \Omega$, $L = 0.4$ H, $C = 25$ mF, $v(0) = 0$, $i(0) = 50$ mA. Find $v(t)$ for $t > 0$.



- where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha = 1/(2RC) = 1/(2x2x25x10^{-3}) = 10$

$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{0.4x25x10^{-3}} = 10$

since $\alpha = \omega_o$, we have a critically damped response. Therefore,

$$s_1 = s_2 = -\alpha:$$

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$

$$v(t) = [(A_1 + A_2 t)e^{-10t}]$$

- Initial conditions $v(0) = V_0$ $\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$

$v(0) = 0 = A_1 + A_2x0 = A_1$, which leads to $v(t) = [A_2 te^{-10t}]$.

$dv(0)/dt = -(v(0) + Ri(0))/(RC) = -2x0.05/(2x25x10^{-3}) = -2$

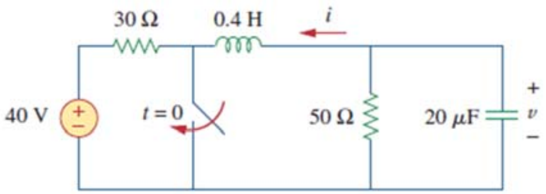
$$v(t) = [(A_1 + A_2 t)e^{-10t}]$$

$$dv/dt = [(A_2 - 10A_2 t)e^{-10t}]$$

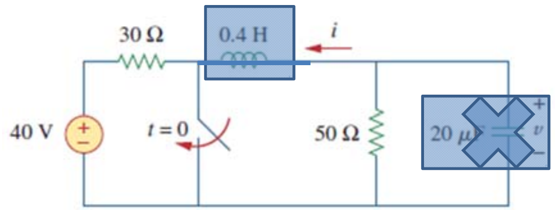
At $t = 0$, $-2 = A_2$ therefore, $v(t) = \mathbf{(-2t)e^{-10t}u(t)\ V}$

Example 3

- Find $v(t)$ for $t > 0$ in the RLC circuit of Fig.



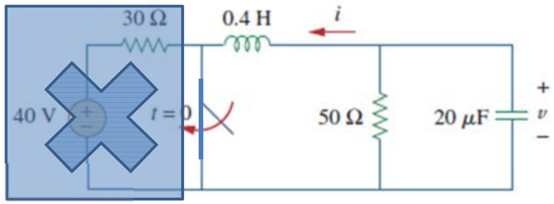
- When t<0 the switch is open; the inductor acts like a short circuit while the capacitor behaves like an open circuit.



- $$v(0) = \frac{50}{30 + 50}(40) = \frac{5}{8} \times 40 = 25 \text{ V}$$

$$i(0) = -\frac{40}{30 + 50} = -0.5 \text{ A}$$

- When t>0 , the switch is closed.



where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4 \times 20 \times 10^{-6}}} = 354$$

Case 1: Overdamped ($\alpha > \omega_0$) $L > 4R^2C.$

$s_{1,2}$ are real: $v(t) = A_1e^{s_1t} + A_2e^{s_2t}$ $\quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o}$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250{,}000 - 124{,}997.6} = -500 \pm 354$$

$$s_1 = -854, \qquad s_2 = -146$$

$$v(t) = A_1e^{-854t} + A_2e^{-146t}$$

- Initial conditions $v(0) = V_0$ $\quad \frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$

At $t = 0$,

$$v(0) = 25 = A_1 + A_2 \quad \Rightarrow \quad A_2 = 25 - A_1$$

$$\frac{dv}{dt} = -854A_1e^{-854t} - 146A_2e^{-146t}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{25 - 50 \times 0.5}{50 \times 20 \times 10^{-6}} = 0$$

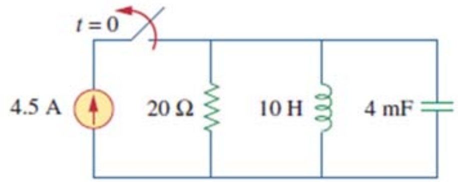
$$0 = 854A_1 + 146A_2$$

$$A_1 = -5.156, \qquad A_2 = 30.16$$

$$v(t) = -5.156e^{-854t} + 30.16e^{-146t}\ \text{V}$$

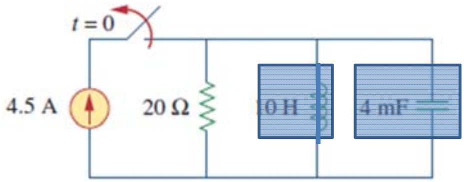
Example 04

Refer to the circuit in Fig. . Find $v(t)$ for $t > 0$.



- For t < 0, the switch is closed. The inductor acts like a short circuit while the capacitor acts like an open circuit. Hence,

  i(0) = 4.5A and v(0) = 0.



where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha = 1/(2RC) = 1/(2x20x4x10^{-3}) = 6.25$

$\omega_o = 1/\sqrt{LC} = 1/\sqrt{10x4x10^{-3}} = 5$

Since $\alpha > \omega_o$, this is an overdamped response.

$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o}$

$= -6.25 \pm \sqrt{(6.25)^2 - 25} = -2.5 \text{ and } -10$

$v(t) = A_1e^{-2.5t} + A_2e^{-10t}$

- Initial conditions $v(0) = V_0$ $\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$

$dv(0)/dt = -(v(0) + Ri(0))/(RC) = -(20x4.5)12.5 = -1125$

But, $dv/dt = -2.5A_1e^{-2.5t} - 10A_2e^{-10t}$

At $t = 0$, $-1125 = -2.5A_1 - 10A_2 = 7.5A_1$ since $A_1 = -A_2$

$A_1 = -150$, $A_2 = 150$

∴, $v(t) = \mathbf{150(e^{-10t} - e^{-2.5t})\ V}$