Lecture (04)
Second Order Circuits I

By:

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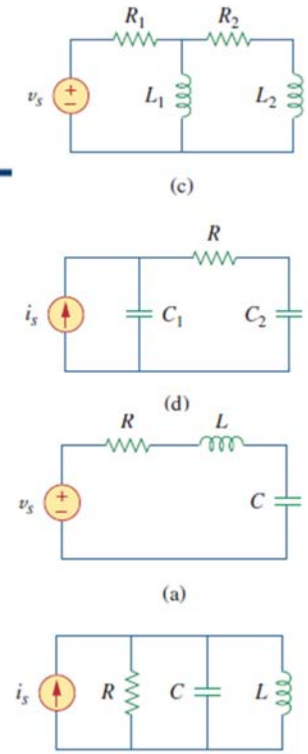
Introduction



- we will consider circuits containing two storage elements. These are known as *second-order* circuits because their responses are described by differential equations that contain second derivatives.
- that a second-order circuit may have two storage elements of different type or the same type

Finding Initial and Final Values

- The goal is to find,

$$v(0),\ i(0),\ dv(0)/dt,\ di(0)/dt,\ i(\infty),\ v(\infty).$$

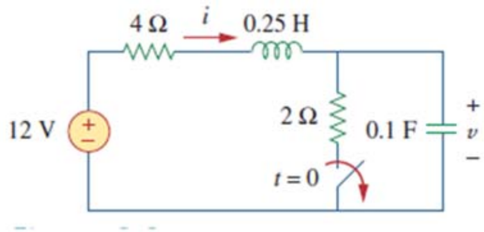
- we must carefully handle the polarity of voltage across the capacitor and the direction of the current through the inductor
- keep in mind that the capacitor voltage is always continuous, and the inductor current is always continuous

$$v(0^+) = v(0^-)$$

$$i(0^+) = i(0^-)$$

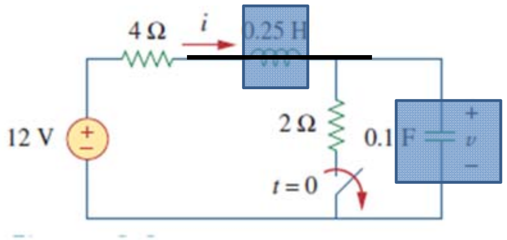
Example 01

The switch in Fig. has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.



- Before t<0, circuit reached steady state
- ohm

$$i(0^-) = \frac{12}{4+2} = 2\text{ A},$$



- Ohm

$$v(0^-) = 2i(0^-) = 4\text{ V}$$

- As the inductor current and the capacitor voltage cannot change suddenly

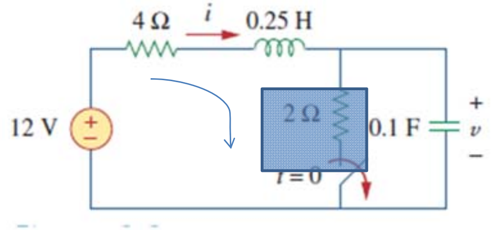
$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

- t>0, switch is open

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

- Capacitor ohm's law

$C\,dv/dt = i_C$, $dv/dt = i_C/C$,



$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{ V/s}$$

- KVL

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

$$v_L(0^+) = 12 - 8 - 4 = 0$$

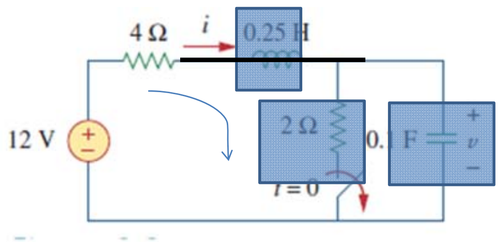
- Coil ohm's law

$L\,di/dt = v_L$, $di/dt = v_L/L$.

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{ A/s}$$

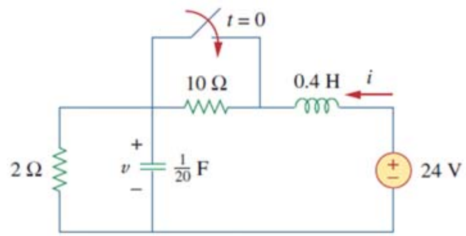
- $t \to \infty$ the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit,

$$i(\infty) = 0\text{ A}, \qquad v(\infty) = 12\text{ V}$$



Example 02

The switch in Fig. was open for a long time but closed at $t = 0$. Determine: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

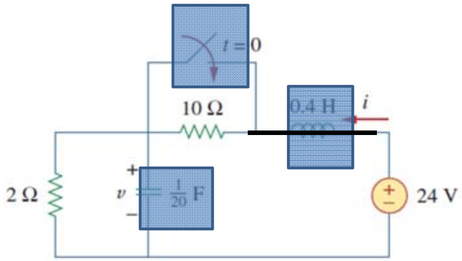


- At t<0 ($t(0^-)$)
- KVL

$$i(0^-) = 24/(2+10) = \mathbf{2\ A},$$

$$v(0^-) = 2i(0^-) = 4\ \text{V}$$

$$\text{hence},\ v(0^+) = v(0^-) = \mathbf{4\ V}.$$



- At t>0 ($t(0^+)$)

$$v(0^+) = v(0^-) = \mathbf{4\ V}.$$

- KVL @ right

$$v_C(0^+) + v_L(0^+) = 24$$

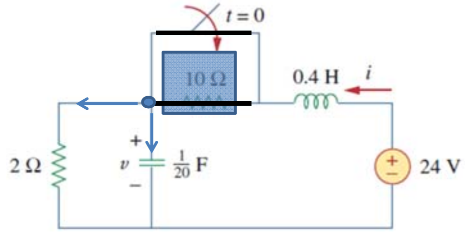
$$24 = 4 + v_L(0+),$$

$$v_L(0+) = 20\ \text{V}$$

- Coil ohm's law

$$L(di/dt) = v_L,\ \text{leads to}\ (di/dt) = v_L/L$$

$$(di(0^+)/dt) = 20/0.4 = \mathbf{50\ A/s}$$



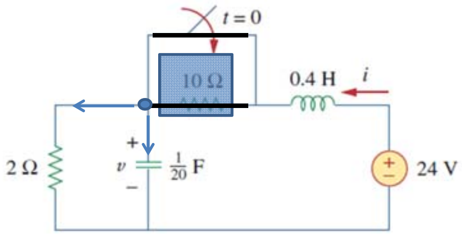
- KCL @ node

$$i(0^+) = i_C(0^+) + v(0^+)/2$$

$$2 = i_C(0^+) + 4/2$$

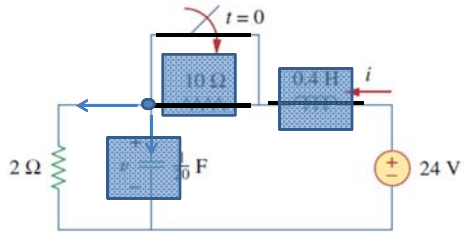
$$i_C(0^+) = 0$$

$$(dv(0^+)/dt) = \mathbf{0\ V/s}$$



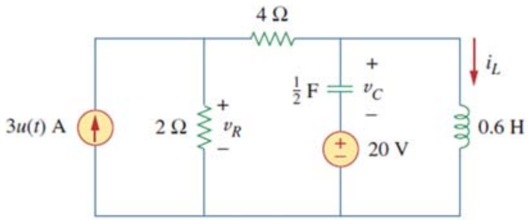
- @t>>0 t→∞
- Ohm's law

$$v(\infty) = \mathbf{24\ V},\ \text{and}\ i(\infty) = \mathbf{12\ A}.$$



Example 03

In the circuit of Fig. , calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.



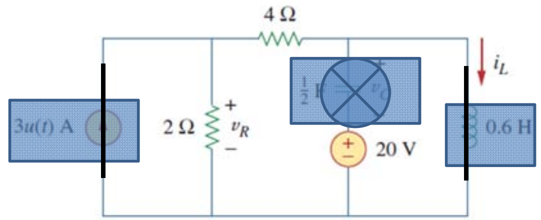
- @ t<0

$3u(t) = 0.$

$i_L(0^-) = 0,$

$v_R(0^-) = 0,$

$v_C(0^-) = -20\text{ V}$



- For t>0

$3u(t) = 3$

- inductor current and capacitor voltage cannot Change immediately

Ω

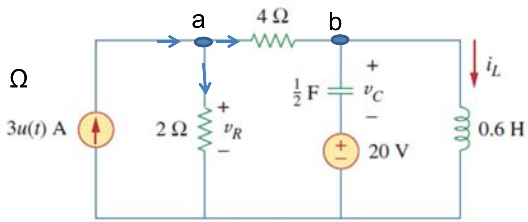
$i_L(0^+) = i_L(0^-) = 0,$

$v_C(0^+) = v_C(0^-) = -20\text{ V}$

- Applying KCL at node *a*

$3 = \frac{v_R(0^+)}{2} + \frac{V_{4\Omega}}{4}$ x-4

$-2VR(0^+) - V_{4\Omega}(0^+) = -12$ →1



- Applying KVL to the middle mesh

$-v_R(0^+) + V_{4\Omega}(0^+) + v_C(0^+) + 20 = 0$

- While $v_C(0^+) = -20\text{ V}$

$-v_R(0^+) + V_{4\Omega}(0^+) = 0$ →2

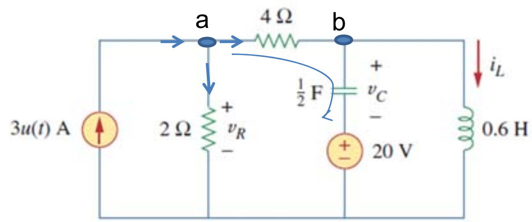
- Add 1,2

$-3VR(0^+) = -12$

$V_R(0^+) = 4\,V$

- Substitute in 2

$V_{4\Omega}(0^+) = 4\,V$



- applying KVL to the right mesh

$$v_L(0^+) - v_C(0^+) - 20 = 0$$

$$v_C(0^+) = -20 \text{ V}$$

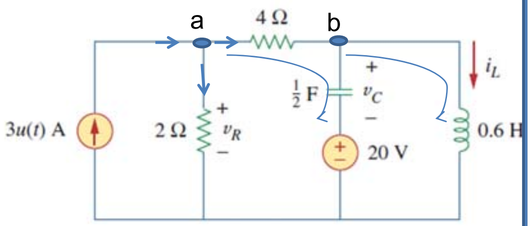
$$v_L(0^+ = 0$$

- Apply Ohms law on coil

$$L\, di_L/dt = v_L$$

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

$$\frac{di_L(0^+)}{dt} = 0$$



- Apply KCL @ b

$$\frac{V_{4\Omega}}{4} = i_C(0^+) + i_L(0^+)$$

- while $V_{4\Omega}(0^+) = 4\,V$
- And $i_L(0^+) = i_L(0^-) = 0,$

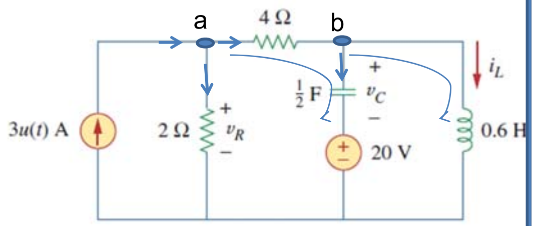
$$i_C(0^+) = 4/4 = 1 \text{ A}.$$

- Ohm's law on capacitor

$$C\, dv_C/dt = i_C$$

$$dv_C/dt = i_C/C.$$

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2 \text{ V/s}$$



To get $dv_R(0^+)/dt$,

- apply KCL to node *a* and obtain

$$3 = \frac{v_R(0^+)}{2} + \frac{V_{4\Omega}}{4}$$

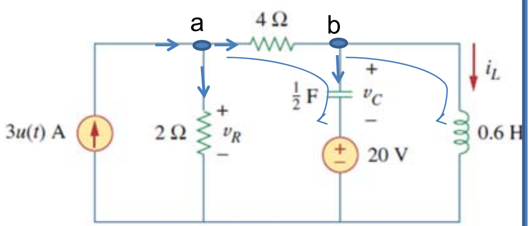
- Taking the derivative of each term

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dV_{4\Omega}(0^+)}{dt} \rightarrow 1$$

- Applying KVL to the middle mesh

$$-v_R(0^+) + V_{4\Omega}(0^+) + v_C(0^+) + 20 = 0$$

- taking the derivative of each term



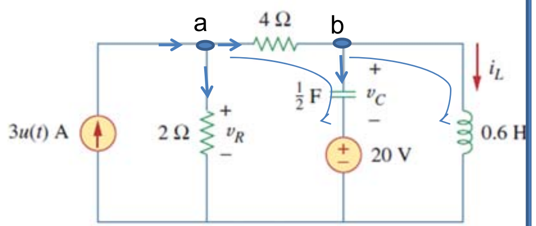
- To find $di_R(0^+)/dt$

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dV_{4\Omega}(0^+)}{dt}$$

$$0 = 2 + \frac{dV_{4\Omega}(0^+)}{dt} - \frac{dv_R(0^+)}{dt}$$

- $\frac{dv_R}{dt} = \frac{2}{3}$
- From ohm $V_R = 2\,IR$

$$\frac{di_R(0^+)}{dt} = \frac{1}{2}\frac{dv_R(0^+)}{dt} = \frac{1}{2}\frac{2}{3} = \frac{1}{3} \text{ A/s}$$

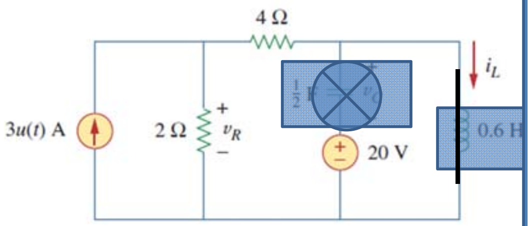


- At t→ ∞

$$i_L(\infty) = \frac{2}{2+4} 3 \text{ A} = 1 \text{ A}$$

$$v_R(\infty) = \frac{4}{2+4} 3 \text{ A} \times 2 = 4 \text{ V},$$

$$v_C(\infty) = -20 \text{ V}$$



Source-Free Series RLC Circuit

$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\, dt = v(0)$

$i(0) = I_0$

- Apply KVL to get

$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0$

- Differentiate again

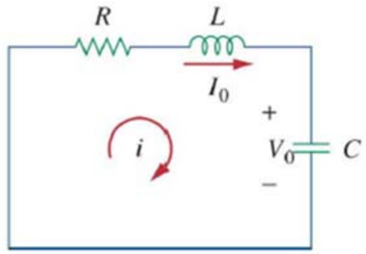
$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$

- Need two initial conditions

$i(0)$ and $\frac{di(0)}{dt}$

- Assume

$i = A\, e^{st}$



- Substitute

$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

- The roots will give us the natural frequencies $s1,2$

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

- General solution will be

$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

The roots s_1 and s_2 are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit;

$\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

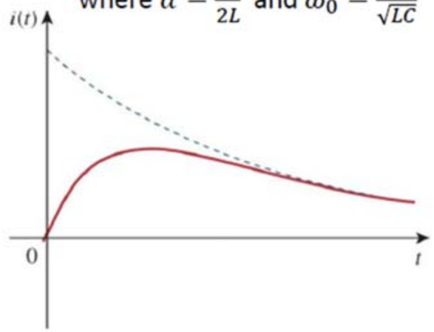
response of the circuit; ω_0 is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s);

and α is the *neper frequency* or the *damping factor*, expressed in nepers per second.

- Case 1: Overdamped $(\alpha > \omega_0)$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$
$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

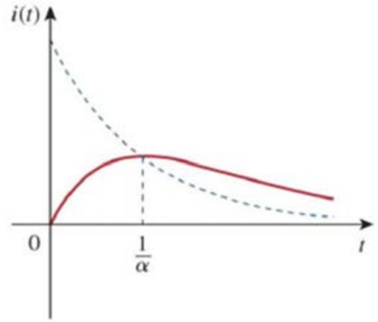


- Case 2: Critically damped $(\alpha = \omega_0)$

$s_1 = s_2 = -\alpha$:

where $\alpha = \frac{R}{2L}$

$$i(t) = (A_1 t + A_2)e^{-\alpha t}$$



$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

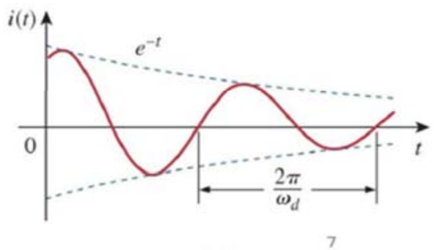
- Case 3: Underdamped $(\alpha < \omega_0)$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$i(t) = e^{-\alpha t}\left(A1 \cos\left(\sqrt{{\omega_0}^2 - \alpha^2}\, t\right) + A2 \sin\left(\sqrt{{\omega_0}^2 - \alpha^2}\, t\right)\right)$$

$s_{1,2} = -\alpha \pm j\omega_d$ where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

ω_d: damping frequency



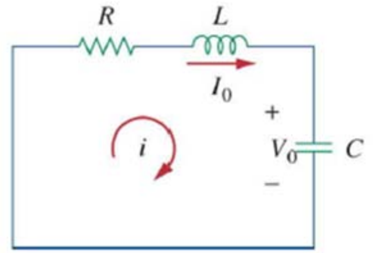
$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- To find A1, and A2,
  use initial conditions I(0), di(0)/dt

At $t = 0$, $i(0) = 1 = A_1 + A2$

$$R\, i(0) + L\frac{di(0)}{dt} + V0 = 0$$

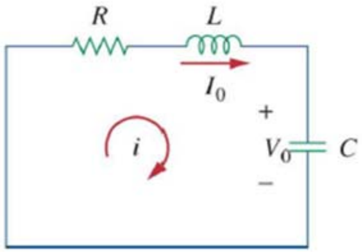
$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$



- Sub di(0)/dt and A1 in i(t) to find A2

Example 05

In Fig. $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?



$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

$R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F.

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5,$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

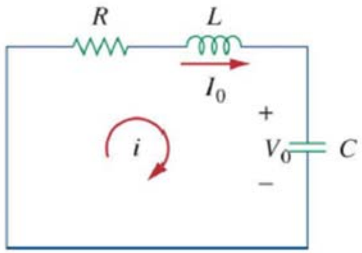
Since $\alpha > \omega_0$, we conclude that the response is overdamped.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

$$s_1 = -0.101, \quad s_2 = -9.899$$

Example 06

If $R = 10\ \Omega$, $L = 5$ H, and $C = 2$ mF in Fig. find α, ω_0, s_1, and s_2. What type of natural response will the circuit have?



where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

- $R = 10\ \Omega$, $L = 5$ H, and $C = 2$ mF

α = R/(2L) = 10/(2x5) = **1**,

$\omega_o = 1/\sqrt{LC} = 1/\sqrt{5\text{x}2\text{x}10^{-2}}$ = **10**

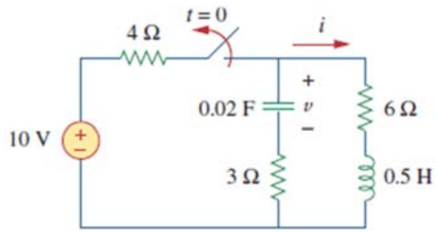
Underdamped $(\alpha < \omega_0)$

$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -1 \pm \sqrt{1 - 100}$ = **–1 ± j9.95**.

$$i(t) = e^{-t}(A1\cos(9.95t) + A2\sin(9.95t))$$

Example 07

Find $i(t)$ in the circuit of Fig.   . Assume that the circuit has reached steady state at $t = 0^-$.

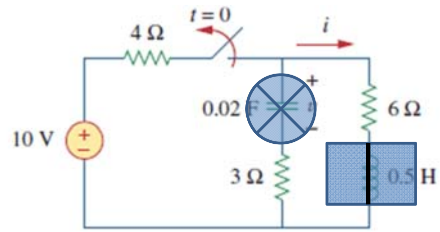


- For t<0 , the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit
- C.D.

$$i(0) = \frac{10}{4+6} = 1 \text{ A},$$

- Ohm

$$v(0) = 6i(0) = 6 \text{ V}$$



where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

- For t>0 the switch is opened and the voltage source is disconnected.
- Req=9 ohm

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9,$$

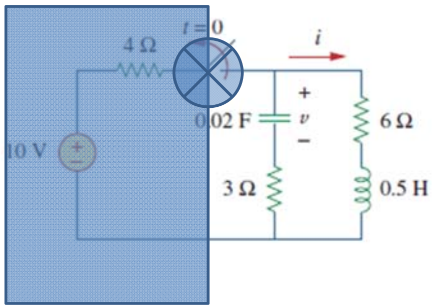
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

the response is underdamped ($\alpha < \omega$);

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

$$s_{1,2} = -9 \pm j4.359$$

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$



- We now obtain A1, A2 using the initial conditions

At $t = 0$, $i(0) = 1 = A_1$

- But $\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$
- Then

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s}$$

- Taking the derivative of $i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

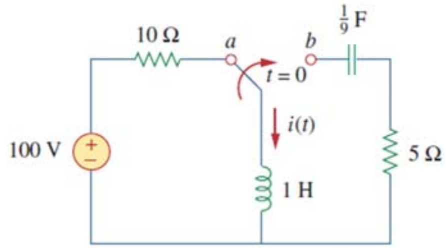
- Substituting di/dt from previous equation at t=0

$$-6 = -9 + 4.359A_2 \quad \Rightarrow \quad A_2 = 0.6882$$

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

Example 08

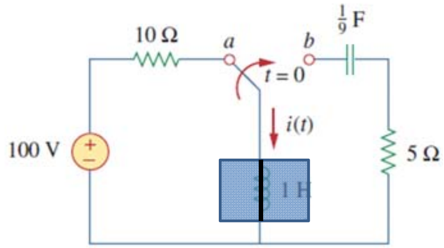
The circuit in Fig. has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.



- For t < 0, the inductor is connected to the voltage source and when the circuit reaches steady state, the inductor acts like a short circuit

$$i(0-) = 50/10 = 5 = i(0+) = i(0)$$

$$0 = v(0-) = v(0+) = v(0).$$



- For t > 0, we have a source-free RLC circuit

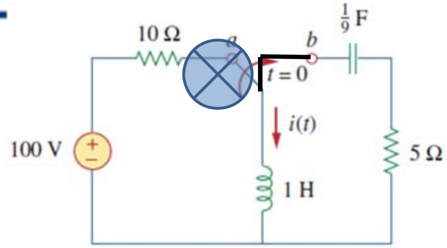
$$\alpha = R/(2L) = 5/(2x1) = 2.5$$

$$\omega_o = 1/\sqrt{LC} = 1/\sqrt{1x\frac{1}{9}} = 3$$

- Since $\alpha < \omega_0$, we have an underdamped case

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -2.5 \pm \sqrt{6.25 - 9} = -2.5 \pm j1.6583$$

$$i(t) = e^{-2.5t}[A_1 \cos 1.6583t + A_2 \sin 1.6583t]$$



- We now determine A1 and A2.

$$i(0) = 10 = A_1$$

- Sub in

$$di(0)/dt = -(1/L)[Ri(0) + v(0)] = -1[5x10+0] = -1[50] = -50$$

- Differentiate

$$i(t) = e^{-2.5t}[A_1 \cos 1.6583t + A_2 \sin 1.6583t]$$

$$di/dt = -2.5\{e^{-2.5t}[A_1 \cos(1.6583t) + A_2 \sin(1.6583t)]\} + 1.6583e^{-2.5t}[-A_1 \sin(1.6583t) + A_2 \cos(1.6583t)]$$

- Sub dt(0)/dt from previous, set t=0

$$-2.5(10) + 1.6583A_2 = -50$$

$$A_2 = -15.076$$

$$\mathbf{i(t) = e^{-2.5t}[10\cos(1.6583t) - 15.076\sin(1.6583t)]\ A}$$

Thanks,..

See you next week (ISA),…