Lecture (03)
First order Circuit (continue,..)

By:

Dr. Ahmed ElShafee

Step Response of an *RC* Circuit

- When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response.*

> The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

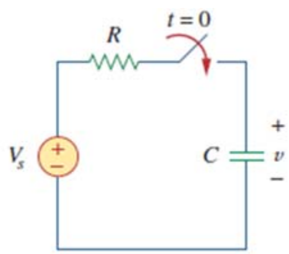
- We assume an initial voltage V_0 on the capacitor . $v(0^-) = v(0^+) = V_0$
- Applying KCL,

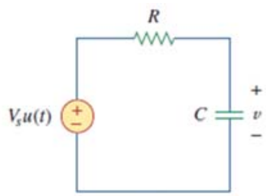
$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

- For t>0

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$





(b)

- Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

- From integration table

$$\int \frac{1}{x}dx = \ln|x|$$

- Integrating both sides and introducing the initial conditions

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

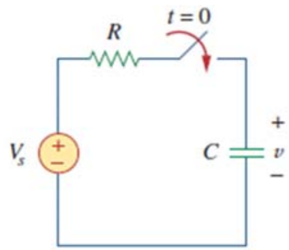
- Taking the exponential of both sides

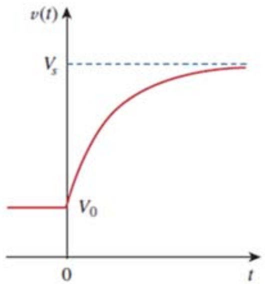
$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$





- capacitor is uncharged initially,

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

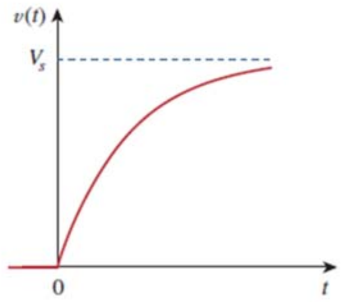
$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

- Calculating current

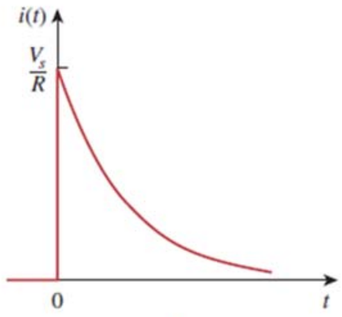
$$i(t) = C\,dv/dt.$$

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$



(a)



(b)

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

- $v(t)$ has two components:
  - “natural response
  - Forced response’’

$$v = v_n + v_f \qquad \text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

$$v_n = V_o e^{-t/\tau}$$

$$v_f = V_s(1 - e^{-t/\tau})$$

- *forced* response because it is produced by the circuit when an external “force’’ (a voltage source in this case) is applied.

- Forced response break to
  - “transient response
  - steady-state response.’’
- Dies:
  - natural response eventually
  - transient component of the forced response, l
- Leaving only
  - the steady state component of the forced response.

- Another way to presents complete response

Complete response = transient response (temporary part) + steady-state response (permanent part)

$$v = v_t + v_{ss}$$

$$v_t = (V_o - V_s)e^{-t/\tau}$$

$$v_{ss} = V_s$$

The **transient response** is the circuit's temporary response that will die out with time.

The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

- $v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

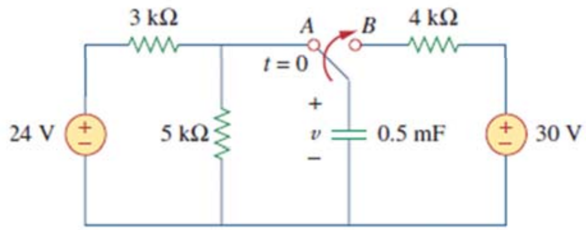
1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

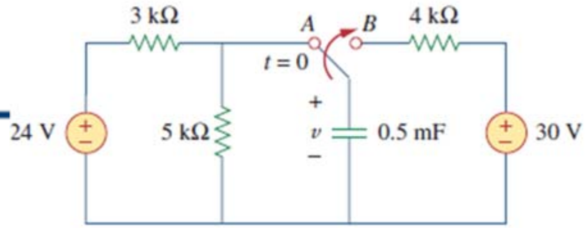
- Note that if the switch changes position at time t_0 instead of *t=0*

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau}$$

Example 01

The switch in Fig. has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



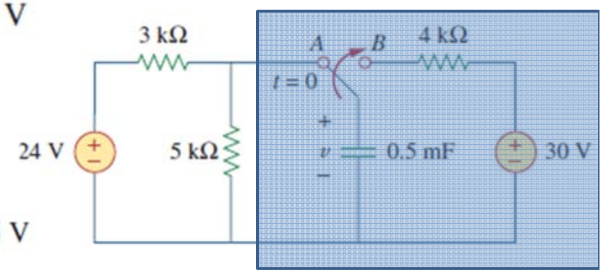


- @ t<0, switch @ A
- voltage division

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

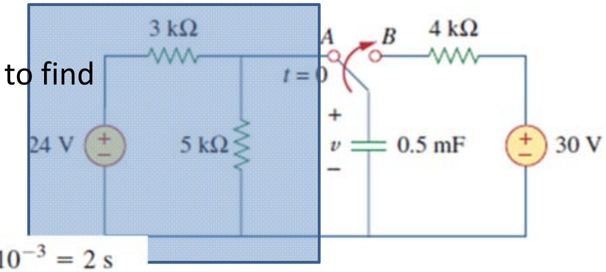
- capacitor voltage cannot change instantaneously

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$



- t>0, switch @ B
- Use thevenin theorem to find Req connected to Cap

$$R_{Th} = 4 \text{ k}\Omega,$$



- Time constant

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

- Cap act ac open circuit @ steady state,

$$v(\infty) = 30 \text{ V}.$$

- as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

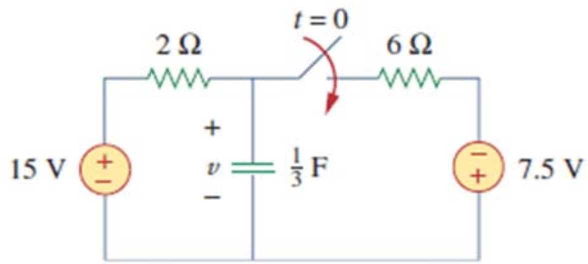
$$v(t) = 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V}$$

At $t = 1$, $v(1) = 30 - 15e^{-0.5} = 20.9 \text{ V}$

At $t = 4$, $v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$

Example 02

Find $v(t)$ for $t > 0$ in the circuit of Fig. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.



- For t<0, switch is open, cap is open

$v(0^-) = v(0^+) = v(0) = 15$

- For t>0
- Find Req

$R_{th} = 2 \| 6 = \frac{3}{2}\Omega,$

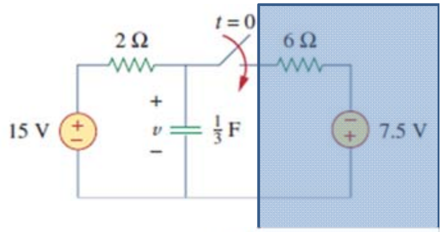
- Find time constant

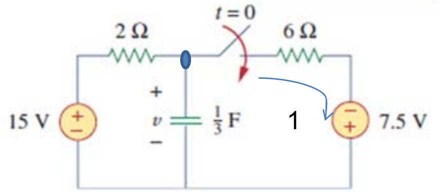
$\tau = R_{th}C = \frac{3}{2} \times \frac{1}{3} = \frac{1}{2}$

- Find v(∞), the current flow through cap=0 (open) then

$[(v(\infty)-15)/2] + [(v(\infty)-(-7.5))/6] = 0$

$v(\infty) = 9.375\ V$





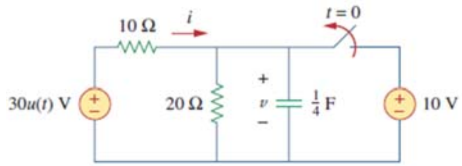
$v(t) = v(\infty) + [v(0) - v(\infty)]\, e^{-t/\tau}$

$v(t) = 9.375 + (15-9.375)e^{-2t}$

At t = 0.5, $v(0.5) = 6.25 + 3.75e^{-1} = 6.25 + 1.3795 = \mathbf{7.63\ V}$

Example 03

In Fig. , the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



- For t<0

$v = 10\ V, \qquad i = -\frac{v}{10} = -1\ A$

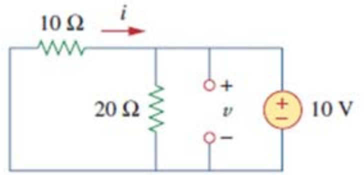
$v(0) = v(0^-) = 10\ V$

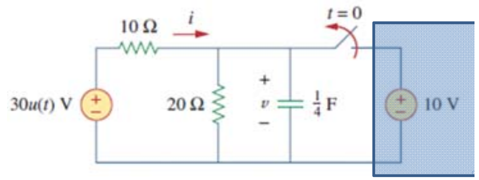
- For t>0
- Find v(∞) Use V.D.

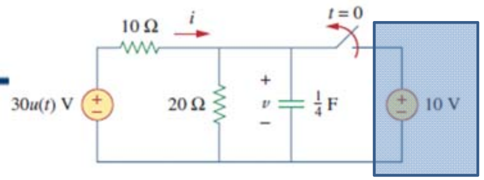
$v(\infty) = \frac{20}{20 + 10}(30) = 20\ V$

- The Thevenin resistance at the capacitor terminals

$R_{Th} = 10 \| 20 = \frac{10 \times 20}{30} = \frac{20}{3}\Omega$







- Time constant

$$\tau = R_{Th}C = \frac{20}{3}\cdot\frac{1}{4} = \frac{5}{3}\text{ s}$$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V}$$

- To find current, that *i* is the sum of the currents through the resistor and the capacitor;

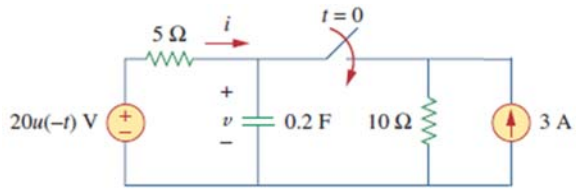
$$i = \frac{v}{20} + C\frac{dv}{dt}$$
$$= 1 + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A}$$

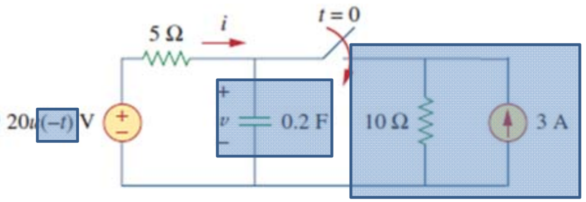
$$v = \begin{cases} 10\text{ V}, & t < 0 \\ (20 - 10e^{-0.6t})\text{ V}, & t \ge 0 \end{cases}$$

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$

Example 04

The switch in Fig. is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

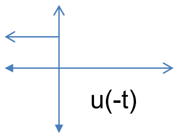




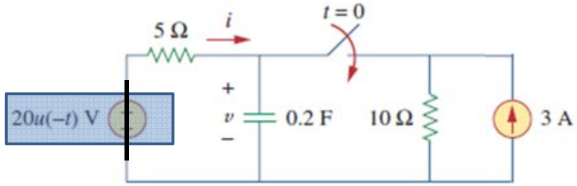
- For t<0, switch is open,

Cap is open

$v(0^-) = v(0^+) = v(0) = 20,$ $\quad$ $i(0) = 0$



- For t>0, switch is on, voltage source =0
- Find v(∞), CD



$$I_{5ohm} = 3 \times \frac{10}{15} = 2$$

- ohm

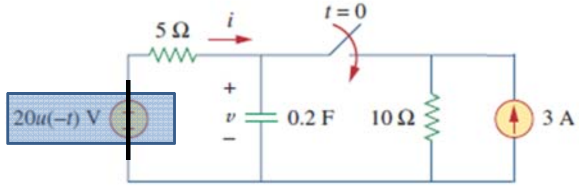
$$V(\infty) = 2 \times 5 = 10\ v$$

$$R_{th} = 5 \| 10 = \frac{10}{3}\Omega$$

- Time constant

$$\tau = R_{th}C = \frac{10}{3}\times 0.2 = \frac{2}{3}$$

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau}$$
$$v(t) = 10 + (20 - 10)e^{-3t/2}$$
$$v(t) = 10\left(1 + e^{-1.5t}\right)$$



- current

$$i(t) = \frac{-v(t)}{5} = -2\left(1 + e^{-1.5t}\right)$$

$$i(t) = \begin{cases} 0 & t < 0 \\ -2\left(1 + e^{-1.5t}\right)\text{A} & t > 0 \end{cases}$$

$$v(t) = \begin{cases} 20\text{ V} & t < 0 \\ 10\left(1 + e^{-1.5t}\right)\text{V} & t > 0 \end{cases}$$

Step Response of an *RL* Circuit

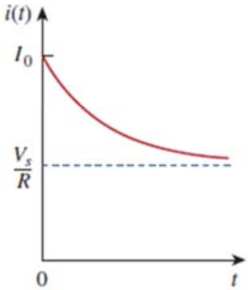
- Complete response = transient response + steady-state response
  (temporary part) (permanent part)

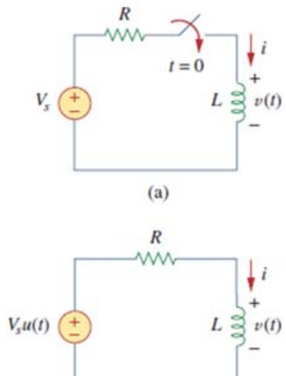
$$i = i_t + i_{ss}$$

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R}$$

$$i_{ss} = \frac{V_s}{R}$$

$$i = Ae^{-t/\tau} + \frac{V_s}{R}$$





(a)

(b)

- At t=0 $I_0 = A + \frac{V_s}{R}$
- we obtain *A* as $A = I_0 - \frac{V_s}{R}$
- Substituting for *A*

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

- This is the complete response of the *RL* circuit

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

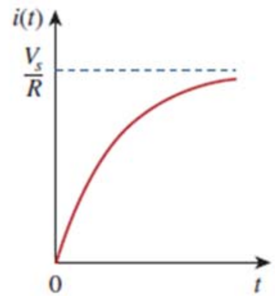
if the switching takes place at time $t = t_0$ instead of $t = 0$,

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

- If I0=0

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

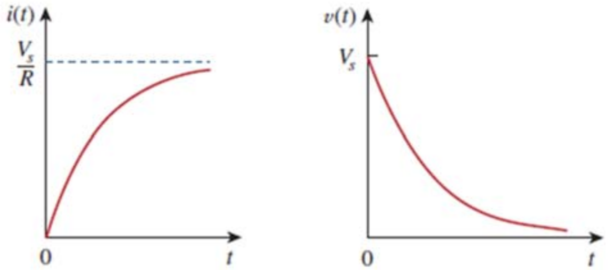
$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$



- The voltage across the inductor is

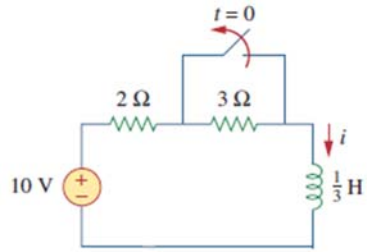
$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

$$v(t) = V_s e^{-t/\tau}u(t)$$



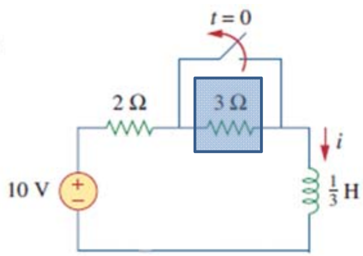
Example 05

Find $i(t)$ in the circuit of Fig. for $t > 0$. Assume that the switch has been closed for a long time.



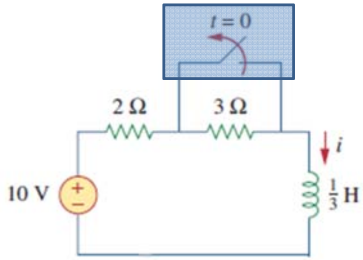
- t<0, switch on, 3 ohm is neglected, coil is short

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$



- t>0, switch is off
- Calculate current using ohm law, coil is short

$$i(\infty) = \frac{10}{2 + 3} = 2 \text{ A}$$



- Calculate Rth

$$R_{\text{Th}} = 2 + 3 = 5 \ \Omega$$

- Calculate time constant

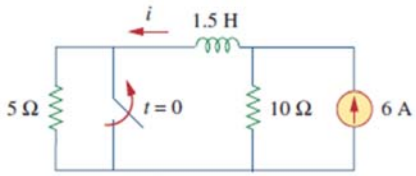
$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

- Complete response

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$
$$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0$$

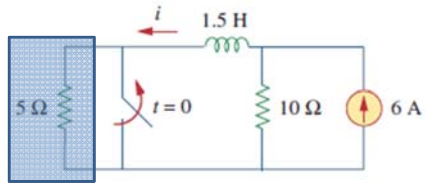
Example 06

The switch in Fig. has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.



- At t<0, the switch is closed so that the 5 ohm resistor is short circuited

$$I(0) = 6\ Amp$$

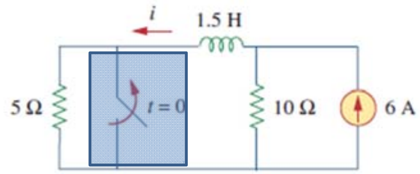


- At t>0, switch is on,
- Find current using CD:

$$I(5ohm) = 6 \times \frac{10}{15} = 4\ Amp$$

- Find Req

$$Rth = 10 + 5 = 15\ ohm$$



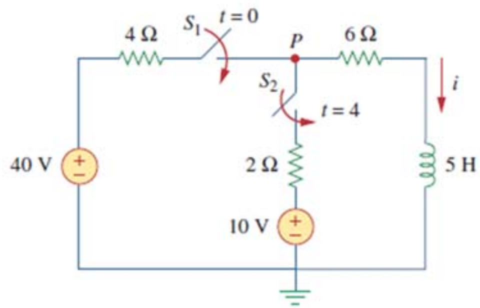
$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau}$$

$$i(t) = 4 + (6 - 4)e^{-10t}$$

$$i(t) = \mathbf{(4 + 2e^{-10t})\ A\ for\ all\ t > 0}$$

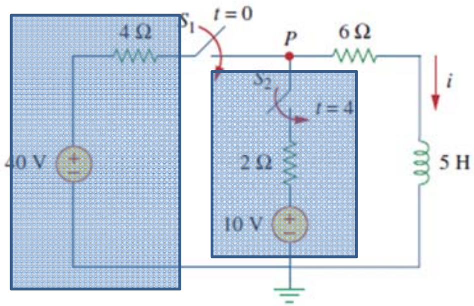
Example 07

At $t = 0$, switch 1 in Fig. is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.



- t<0

$$i(0^-) = i(0) = i(0^+) = 0$$

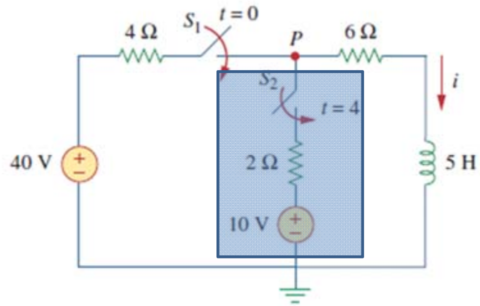


- 0<t<4

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A},$$

$$R_{Th} = 4 + 6 = 10\,\Omega$$

$$\tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2}\text{ s}$$



$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 4 + (0 - 4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \le t \le 4$$

For $t \ge 4$,

$$i(t) = 4(1 - e^{-2t}) \text{ A},$$

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

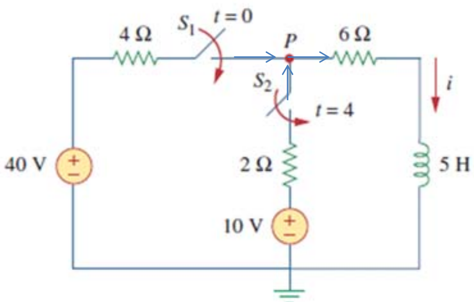
- Use KCL @ p

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6}$$

$$v = \frac{180}{11}\text{ V}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727 \text{ A}$$

$$R_{Th} = 4 \,\|\, 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\,\Omega$$



$$\tau = \frac{L}{R_{Th}} = \frac{5}{\frac{22}{3}} = \frac{15}{22}\text{ s}$$

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \ge 4$$

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \ge 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \le 0 \\ 4(1 - e^{-2t}), & 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \ge 4 \end{cases}$$

At $t = 2$,

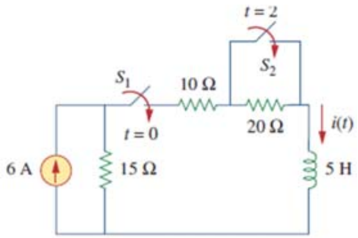
$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ A}$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-1.4667} = 3.02 \text{ A}$$

Example 08

Switch S_1 in Fig. is closed at $t = 0$, and switch S_2 is closed at $t = 2$ s. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

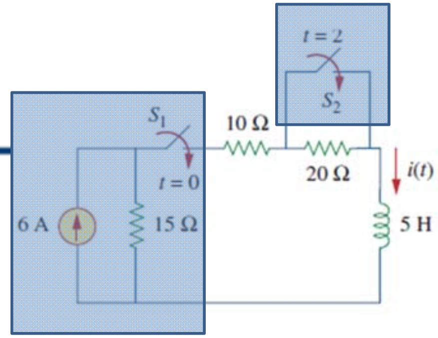


- At t<0, S1, S1 are open

$i(0^-) = 0$

- At 0<t<2, S1 close, S2 open
- Calculate current using CD

$$I(\infty) = 6 \times \frac{15}{10 + 15 + 20} = 2\ Amp$$



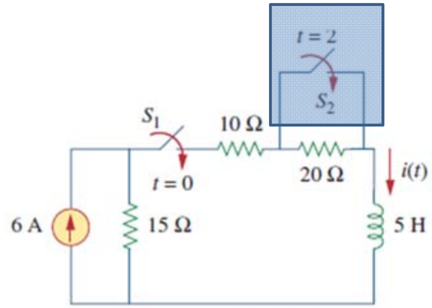
$$R_{th} = 45\,\Omega$$

$$\tau = \frac{L}{R_{th}} = \frac{5}{45} = \frac{1}{9}$$

$$i(t) = i(\infty) + \left[i(0) - i(\infty) \right] e^{-t/\tau}$$

$$i(t) = 2 + (0 - 2)e^{-9t}$$

$$i(t) = 2(1 - e^{-9t})\ A$$



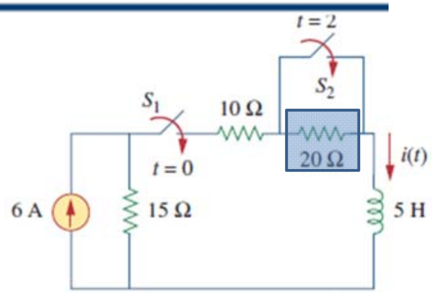
- At t>2,

$i(t) = 2(1 - e^{-9t})\ A$

$i(2^+) = i(2^-) = 2(1 - e^{-18}) \cong 2$

- Calculate I(∞), using CD

$$I(\infty) = 6 \times \frac{15}{15 + 10} = 3.6\ Amp$$



$$R_{th} = 25\,\Omega, \quad \tau = \frac{5}{25} = \frac{1}{5}$$

$$i(t) = i(\infty) + \left[i(2^+) - i(\infty) \right] e^{-(t-2)/\tau}$$

$$i(t) = 3.6 + (2 - 3.6)e^{-5(t-2)}$$

$$i(t) = 3.6 - 1.6e^{-5(t-2)}$$

$$\text{Thus,} \quad i(t) = \begin{cases} \mathbf{0} & \mathbf{t < 0} \\ \mathbf{2(1 - e^{-9t})\ A} & \mathbf{0 < t < 2} \\ \mathbf{3.6 - 1.6e^{-5(t-2)}\ A} & \mathbf{t > 2} \end{cases}$$

At $t = 1$, $\quad i(1) = 2(1 - e^{-9}) = \mathbf{1.9997\ A}$

At $t = 3$, $\quad i(3) = 3.6 - 1.6e^{-5} = \mathbf{3.589\ A}$

Thanks,..

See you next week (ISA),…