Lecture (02)
First order Circuit (continue,..)

By:

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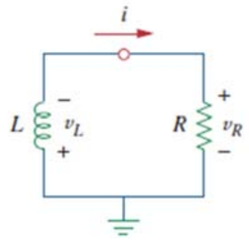
The Source-Free *RL* Circuit

- Consider the series connection of a resistor and an inductor
- Goal is to determine the current flow through the inductor i(t),
- At t=0, assume the initial current = I_0



$$i(0) = I_0$$

- with the corresponding energy stored in the inductor as

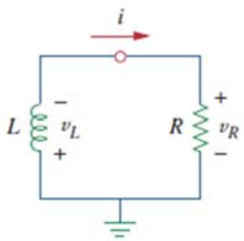
$$w(0) = \frac{1}{2} L I_0^2$$

- Applying KVL around the loop

$$v_L + v_R = 0$$

- Ohm:

$$v_L = L\, di/dt \text{ and } v_R = iR.$$



- Then:

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0$$

- Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

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- From integration table: $\int \frac{1}{x} dx = \ln |x|$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

- so

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$

- Taking the powers of *e* $\quad i(t) = I_0 e^{-Rt/L}$
- This shows that the natural response of the *RL* circuit is an exponential decay of the initial current

- the time constant for the *RL* circuit is

$$\tau = \frac{L}{R}$$

$$i(t) = I_0 e^{-t/\tau}$$

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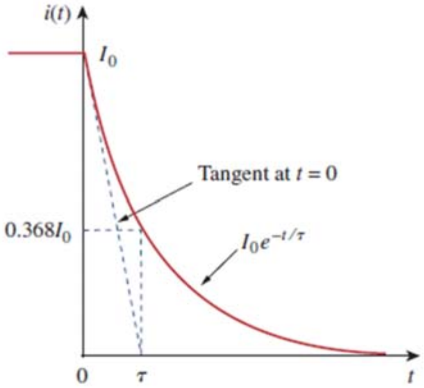


- voltage across the resistor $v_R(t) = iR = I_0 R e^{-t/\tau}$
- The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

- The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t I_0^2 R\, e^{-2\lambda/\tau}\, d\lambda$$

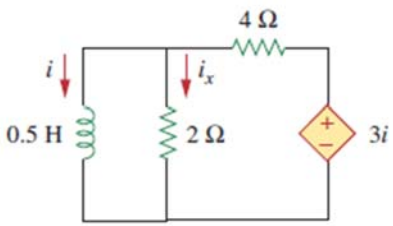
$$w_R(t) = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau}\Big|_0^t$$

- But $\tau = \frac{L}{R}$
- then $w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$,

Example 01

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig.

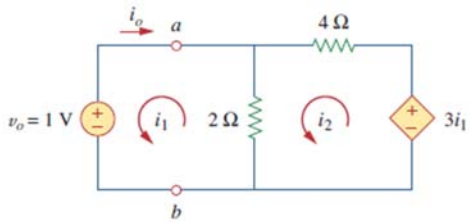


- Firstly, The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with V0=1V at the inductor terminals *a-b*,
- Applying KVL to the two loops results in
- Left loop:

$$2(i1 - i2) + 1 = 0$$

$$i1 - i2 = -\frac{1}{2} \quad \text{x6}$$

$$6i1 - 6i2 = 3$$

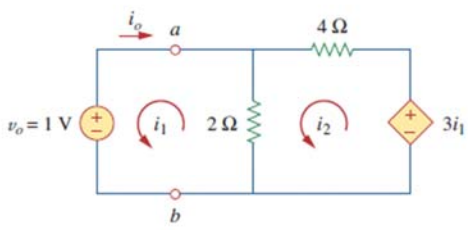


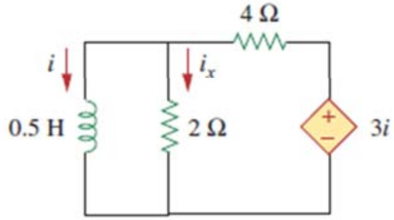
- Right loop

$$2(i2 - i1) - 3i1 + 4i2 = 0$$
$$-5i1 + 6i2 = 0$$

- Add both equations

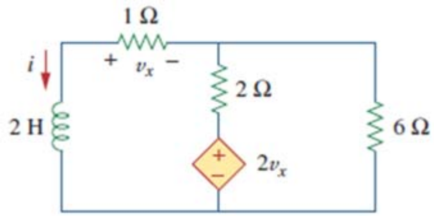
$$i1 = -3$$
$$i1 = -3\ Amp$$
$$i0 = 3\ Amp$$
$$Rth = Req = \frac{1}{3} Ohm$$
$$\tau = \frac{L}{Req} = \frac{3}{2} s$$
$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\ \text{A}, \qquad t > 0$$





Example 02

Find i and v_x in the circuit of Fig. 7.15. Let $i(0) = 12$ A.



- Ohm

$$Vx = 1 \times I$$

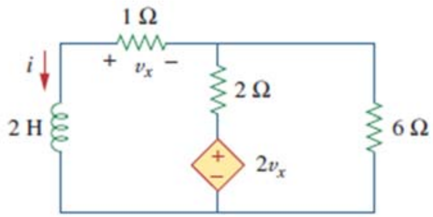
- KCL @1

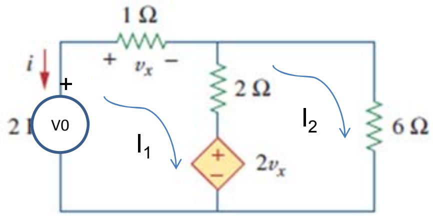
$$I1 + 2(I1 - I2) + 2Vx - 1 = 0$$
$$I1 + 2I1 - 2I2 + 2I1 = 1$$
$$5I1 - 2I2 = 1 \quad \text{x4}$$
$$20\, I1 \ - 8\, I2 = 4 \rightarrow 1$$

- KCL@2

$$2(I2 - I1) - 2Vx + 6I2 = 0$$
$$2I2 - 2I1 - 2I1 + 6I2 = 0$$
$$-4I1 + 8I2 = 0 \rightarrow 2$$

- Add 1,2





$$16 - I1 = 4$$
$$I1 = \frac{4}{16} = \frac{2}{12} = \frac{1}{4} Amp$$

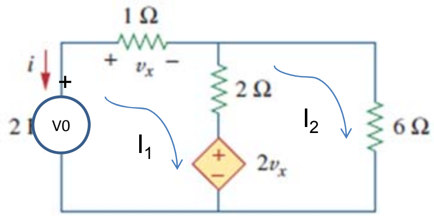
$$I0 = I1 = 1/4 Amp$$
$$Req = \frac{V0}{I0} = 4\ Ohm$$

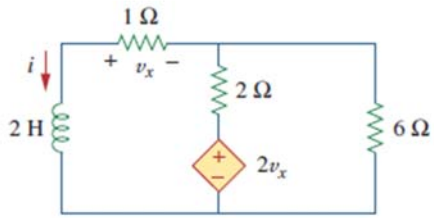
- coil

$$I(t) = I0\ e^{-tR/L} = 12\ e^{-4t/2}$$
$$= 12\ e^{-2t}$$

- ohm

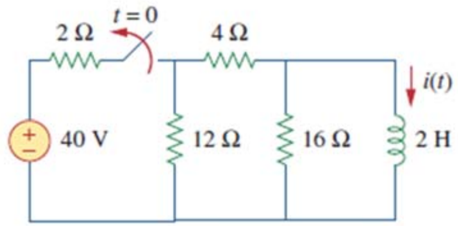
$$Vx = -1 \times I(t) = -12e^{-2t}$$





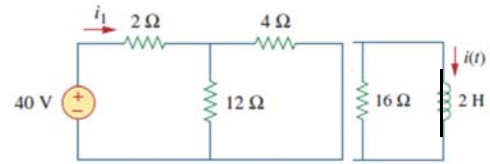
Example 03

The switch in the circuit of Fig. has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.



- Step 1, to find i(0)
- @t<0 the switch is closed, and the inductor acts as a short circuit to dc. 16 ohm resistance is short circuit
- 4 in parallel with 12 → 3 ohm
- 3 in series with 2 → 5 ohm
- Ohm :

$$i1 = \frac{40}{5} = 8\ amp$$

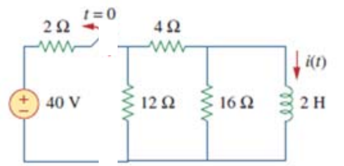


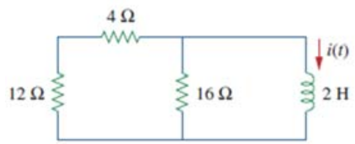
- C.D.

$$i(0) = 8\ \times \frac{12}{4+12} = 6\ Amp$$

- @t>0 the switch is open and the voltage source is disconnected. We now have the source-free *RL*
- 12 in series with 4 →16 ohm
- 16 in series with 16 →8 ohm

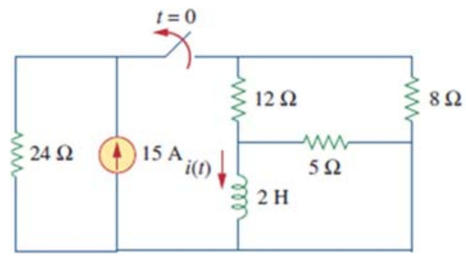
$$i(t) = i0\ e^{-tR/L} = 6\ e^{-8t/2} = 6e^{-4t}$$





Example 04

For the circuit in Fig. , find $i(t)$ for $t > 0$.

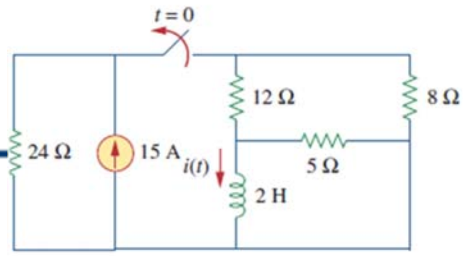


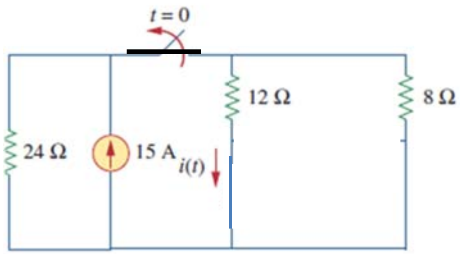
- At t<0, switch is closed, coil is short

$$8 \text{ in parallel with } 24 \rightarrow 6 \text{ ohm}$$

- Current division

$$I_{12ohm} = I(0) = 15 \times \frac{6}{12+6} = 5 \text{ Amp}$$



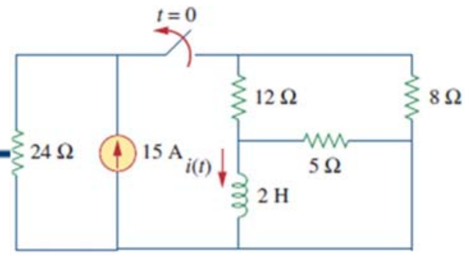


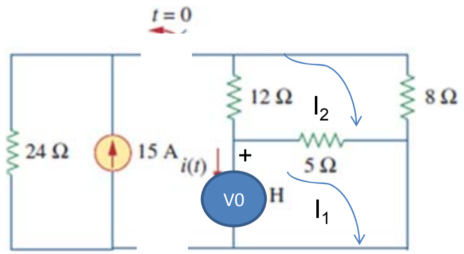
- At t>0, switch is opened
- Use thevenin theorem to find Req as seen by coil, by replacing coil with voltage source V0=1 volt
- KVL@1

$$-1 + 5(I1 - I2) = 0$$

$$5I1 - 5I2 = 1 \quad \text{x5}$$

$$25I1 - 25I2 = 5 \rightarrow 1$$





- KVL@2

$$20I2 + 5(I2 - I1) = 0$$

$$20I2 + 5I2 - 5I1 = 0$$

$$-5I1 + 25I2 = 0 \rightarrow 2$$

- Add 1,2

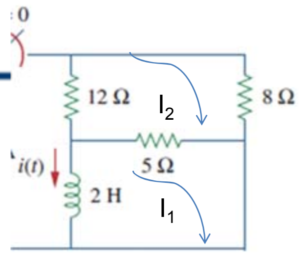
$$20I1 = 5$$

$$I1 = \frac{5}{20} = \frac{1}{4}$$

- Req

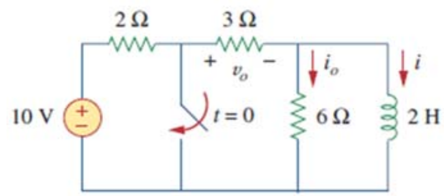
$$Req = Rth = \frac{V0}{I0} = 4 \text{ ohm}$$

$$I(t) = I0e^{-tR/L} = 5e^{-4t/2} = 5e^{-2t}$$



Example 05

In the circuit shown in Fig. , find i_o, v_o, and i for all time, assuming that the switch was open for a long time.



$$i_o(t) = \begin{cases} 0 \text{ A}, & t < 0 \\ -\frac{2}{3}e^{-t} \text{ A}, & t > 0 \end{cases}, \quad v_o(t) = \begin{cases} 6 \text{ V}, & t < 0 \\ 4e^{-t} \text{ V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2 \text{ A}, & t < 0 \\ 2e^{-t} \text{ A}, & t \geq 0 \end{cases}$$

- At t<0, switch is opened, coil is shorted

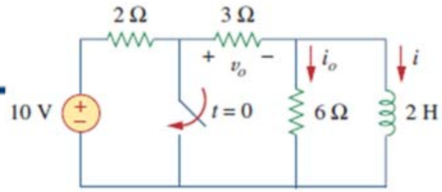
$$I0(0) = 0$$

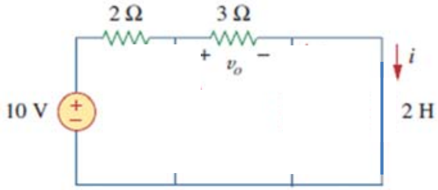
- V.D.

$$V0(0) = 10 \times \frac{3}{2+3} = \frac{30}{5} = 6\,Volt$$

- Ohm

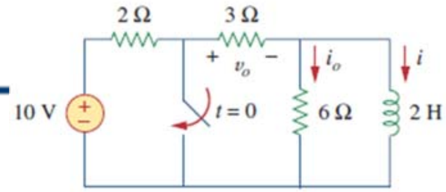
$$i(0) = \frac{10}{5} = 1\,Amp$$

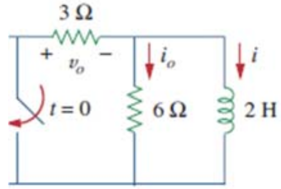




- @ t>0 Switch is closed, so the 10v supply and 2 ohm resistor is independet circuit, has nothing to do with the rest of circuit that contains coil under consideration
- Use thevenin theorem

$$Rth = Req = 3\ in\ parallel\ with\ 6$$
$$= \frac{3 \times 6}{3+6} = \frac{18}{9} = 2\ ohm$$
$$i(t) = i(0)e^{-tR/L} = 2\,e^{-2t/2}$$
$$= 2e^{-t}$$



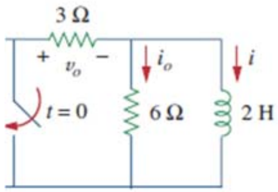


- Ohm

$$Vl(t) = \frac{Ldi}{dt} = 4\frac{d}{dt}e^{-t} = 4e^{-t}\ volt$$
$$V0(t) = Vl = -4e^{-t}\ volt$$

- Ohm

$$I0(t) = \frac{V0}{6} = -\frac{2}{3}e^{-t}\ Amp$$



Summery

<u>*RC*</u>

$$V = V0\ e^{-t/RC}$$
$$\tau = RC$$
$$V = V0\ e^{-t/\tau}$$

$$p(t) = vi_R = \frac{V_0^2}{R}e^{-2t/\tau}$$
$$w_R(t) = \frac{1}{2}CV_0^2(1 - e^{-2t/\tau}),$$

<u>*RL*</u>

$$I = I0e^{-tR/L}$$
$$\tau = \frac{L}{R}$$
$$I = I0e^{-t/\tau}$$

$$p = v_R i = I_0^2 Re^{-2t/\tau}$$
$$w_R(t) = \frac{1}{2}L\,I_0^2(1 - e^{-2t/\tau})$$

Singularity Functions

- Singularity functions (also called *switching functions*) are good approximations to the switching signals that arise in circuits with switching operations.

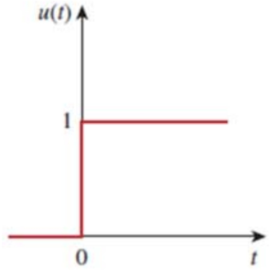
unit step function

1. *unit step*, the *unit impulse*, and the *unit ramp* functions.

- The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

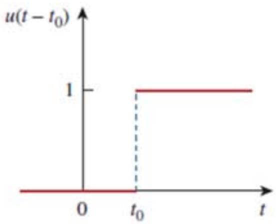
- The unit step function is undefined at where it changes abruptly from 0 to 1.



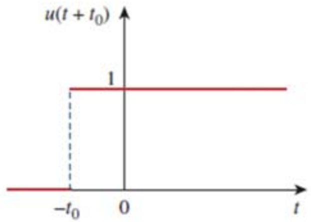
- The unit step function may delayed or stepped by t_0.

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$



(a)



(b)

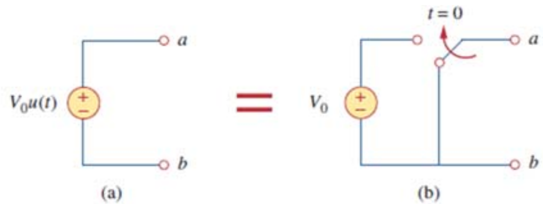
- We use the step function to represent an abrupt (sudden) change in voltage or current, like the changes that occur in the circuits of control systems and digital computers

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

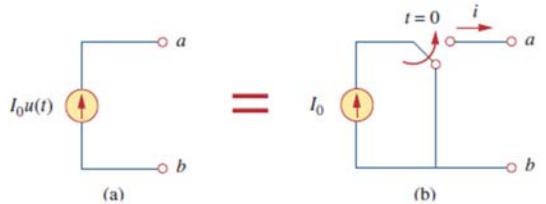
- may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0)$$

- If we let $t=t_0$, then $v(t)$, is simply the step voltage $v_0u(t)$.



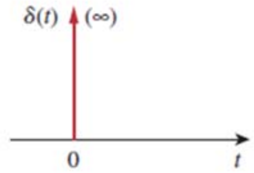
- If we let $t=t_0$, then $i(t)$, is simply the step current $i_0u(t)$.



Unit Impulse function (delta)

The derivative of the unit step function $u(t)$ is the *unit impulse function* $\delta(t)$, which we write as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$



- The unit impulse function—also known as the *delta* function—

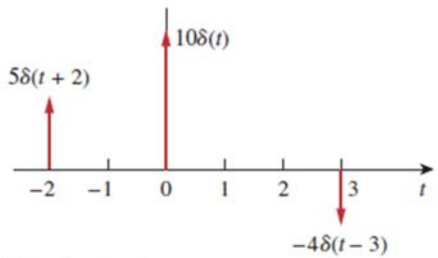
The **unit impulse function** $\delta(t)$ is zero everywhere except at $t = 0$, where it is undefined.

- Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources.
- the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.
- It may be visualized as a very short duration pulse of unit area

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$

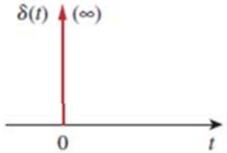
where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$.

- The unit area is known as the *strength* of the impulse function.
- When an impulse function has a strength other than unity, the area of the impulse is equal to its strength.
- For example 10 $\delta(t)$, an impulse function has an area of 10.



- To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)\,dt$$



where $a < t_0 < b$. Since $\delta(t - t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t - t_0)\,dt = \int_a^b f(t_0)\delta(t - t_0)\,dt$$

$$= f(t_0)\int_a^b \delta(t - t_0)\,dt = f(t_0)$$

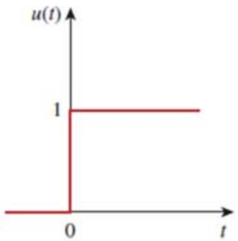
- This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs.
- This is a highly useful property of the impulse function known as the *sampling*
- Special case when t_0=0.

$$\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0)$$

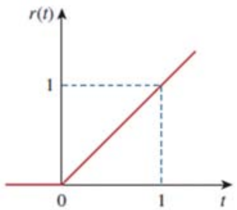
unit ramp function

- Integrating the unit step function results in the *unit ramp function* we write

$$r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda = tu(t)$$



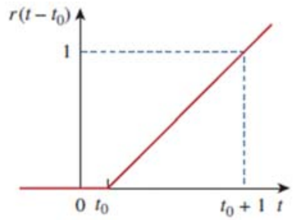
$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$



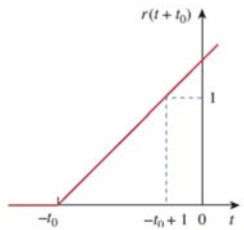
The **unit ramp function** is zero for negative values of *t* and has a unit slope for positive values of *t*.

- For the delayed unit ramp function

$$r(t - t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$



$$r(t + t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases}$$



- We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

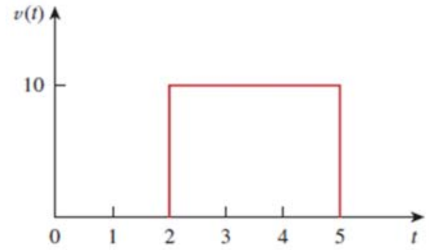
$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

- or by integration as

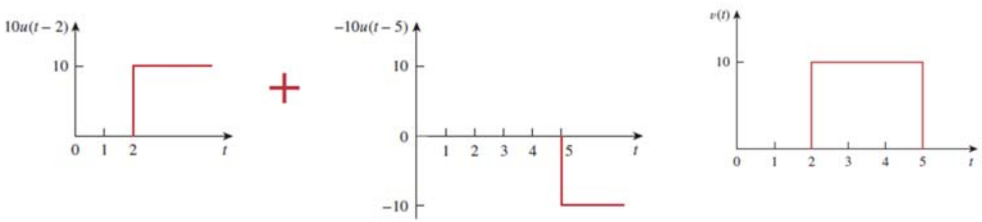
$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda, \qquad r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

Example 06

Express the voltage pulse in Fig. in terms of the unit step. Calculate its derivative and sketch it.



switches on at $t = 2$ s and switches off at $t = 5$ s. It consists of the sum of two unit step functions



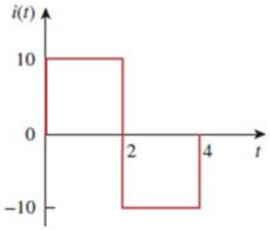
$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

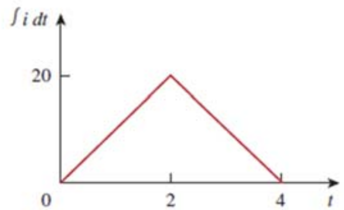
- Taking the derivative of this gives

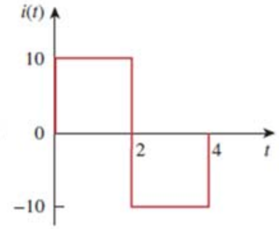
$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

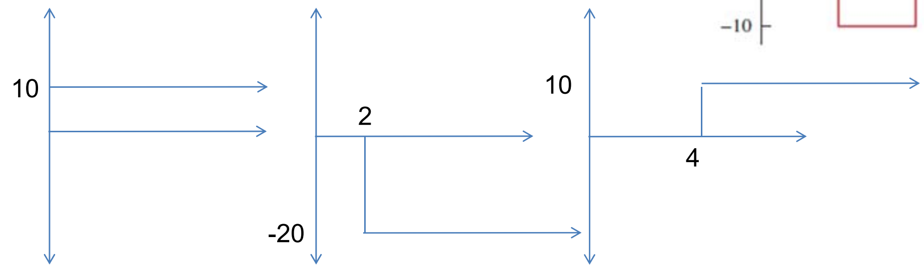
Example 07

Express the current pulse in Fig. in terms of the unit step. Find its integral and sketch it.

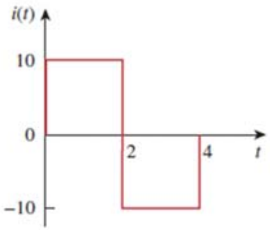








$$f(t) = 10\,u(t) - 20\,u(t-2) + 10\,u(t-4)$$



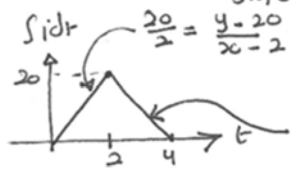
$$r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

$$f(t) = 10\,u(t) - 20\,u(t-2) + 10\,u(t-4)$$

$$f'(t) = 10r(t) - 20\,r(t-2) + 10r(t-4)$$

Example 2
part 2

Slope $= \frac{y - y_1}{x - x_1}$



$\frac{20}{2} = \frac{y-20}{x-2} = 10 = \frac{y-20}{x-2}$ ∴ $10x - 20 = y - 20$

$10t - 20 = \int i\,dt - 20$

$\int i\,dt = 10t$

$\frac{20}{-2} = \frac{y-20}{x-2} = -10, \frac{y-20}{x-2}$

$-10t - 20 = \int i\,dt - 20$

$\int i\,dt = -10t$

$$\int i(t) = \begin{cases} 10t & 0 < t < 2 \\ -10t & 2 < t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$\int i(t) = 10t\,[u(t) - u(t-2)] - 10t\,[(u(t-2) - u(t-4)]$

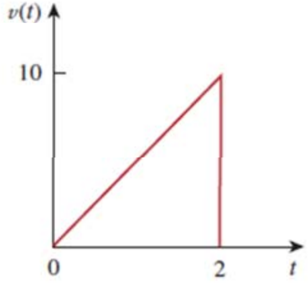
$10t\,[u(t) - u(t-2) - u(t-2) + u(t-4)]$

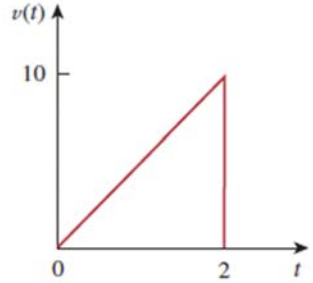
but $u(t)t = r(t)$

$= 10r(t) - 2r(t-2) + 10r(t-4)$

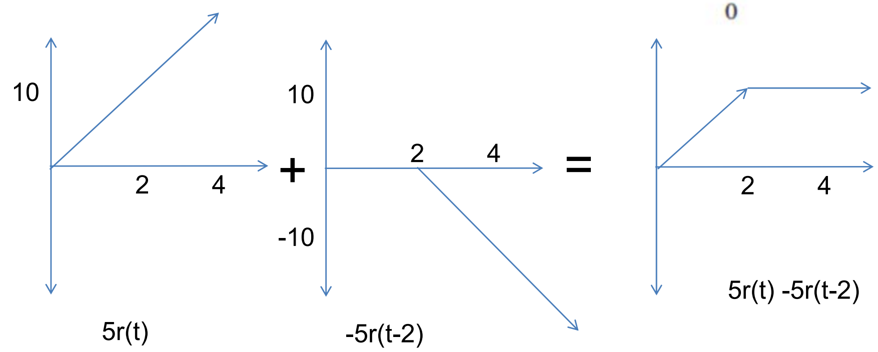
Example 08

Express the *sawtooth* function shown in Fig. in terms of singularity functions.





- Method 1 , addition



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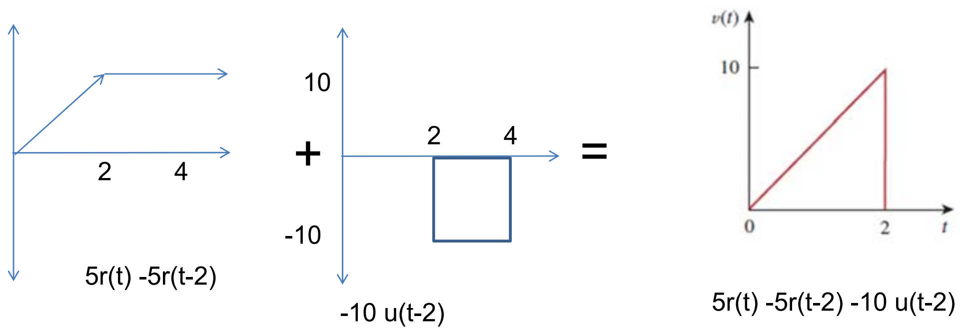
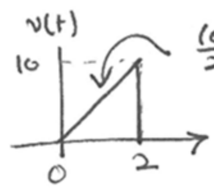


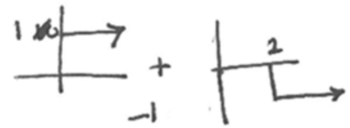
example 03

method 2



$\frac{10}{2} = 5 = \frac{V(t)-0}{t-0} \therefore v(t) = 5t$

$v(t) = \begin{cases} 5t & 0<t<2 \\ 0 & \text{elsewhere} \end{cases}$



$v(t) = 5t\,[u(t) - u(t-2)]$

$= 5t\,u(t) - 5t\,u(t-2)$

$= 5t\,u(t) - 5(t-2+2)\,u(t-2)$

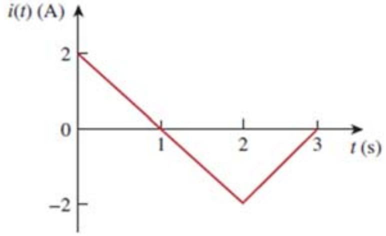
$= 5t\,u(t) - 5(t-2)u(t-2) - 10\,u(t-2)$

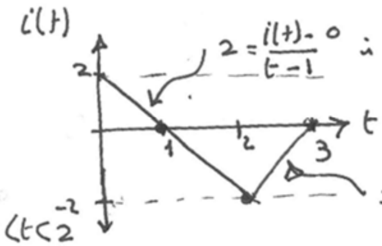
but $t\,u(t) = r(t)$

$= 5r(t) - 5\,r(t-2) - 10\,u(t-2)$

Example 09

- Refer to Fig Express i(t) in terms of singularity functions.





$2 = \frac{i(t) - 0}{t - 1} \therefore i(t) = 2t - 2$

$\frac{-2}{2-3} = \frac{-2}{-1} = 1 = \frac{i(t) - 0}{t - 3}$

$i(t) = 2t - 6$

$$i(t) = \begin{cases} 2t-2 & 0<t<2 \\ 2t-6 & 2<t<3 \\ 0 & \text{elsewhere} \end{cases}$$

$$i(t) = (2t-2)\left[u(t) - u(t-2)\right] + (2t-6)\left[u(t-2) - u(t-3)\right]$$