Lecture (01)
First Order Circuits

By:

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Revision

Capacitors

$$i = \frac{dq}{dt} \qquad q = Cv \qquad C = \frac{\epsilon A}{d}$$

$$i = C\frac{dv}{dt}$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0.

$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}$$

- Notes:

$$i = C\frac{dv}{dt}$$

when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus: A capacitor is an open circuit to dc

- The voltage on a capacitor cannot change suddenly,

Why? huge change ΔV in t=0 → I=∞ , which is physically impossible

Inductors

$$L = \frac{N^2 \mu A}{\ell}$$

$$v = L\frac{di}{dt}$$

$$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$$

$$w = \frac{1}{2}Li^2$$

Notes

$$v = L\frac{di}{dt}$$

- the voltage across an inductor is zero when the current is constant
- An inductor acts like a short circuit to dc.
- The current through an inductor cannot change instantaneously
- The current on a inductor cannot change suddenly,

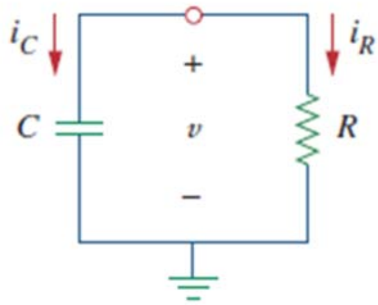
Why? huge change Δi in t=0 → v=∞ , which is physically impossible

Introduction

- we have considered the three passive elements (resistors, capacitors, and inductors)
- we are prepared to consider circuits that contain various combinations of two or three of the passive elements.
- We shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor.
- These are called *RC* and *RL* circuits, respectively.
- the analysis of *RC* and *RL* circuits by applying
- Kirchhoff's laws, as we did for resistive circuits.
- The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to *RC* and *RL* circuits produces differential equations

- The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order.

The Source-Free *RC* Circuit



- A source-free *RC* circuit occurs when its dc source is suddenly disconnected.
- The energy already stored in the capacitor is released to the resistors.
- Our objective is to determine the circuit response,
- we assume to be the voltage V(t), across the capacitor

$$v(0) = V_0 \ \rightarrow \mathrm{t} = 0$$

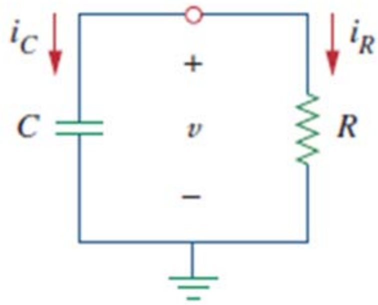
$$w(0) = \frac{1}{2} C{V_0}^2 \ \rightarrow t = 0$$

- KCL :

$$i_C + i_R = 0$$

- Ohm:

$$i_R = v/R$$
$$i_c = C\frac{dv}{dt}$$



- Substitute:

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$
$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

- Rearrange

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

- From integration table: $\int \frac{1}{x}dx = \ln|x|$ $\qquad \frac{dv}{v} = -\frac{1}{RC}dt$
- Solving differential equation by integrating both sides

$$\ln v = -\frac{t}{RC} + k$$

- Taking power e of both sides

$$V(t) = e^{-\frac{t}{RC}} e^{k}$$

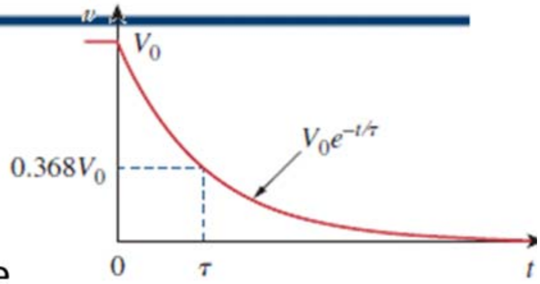
- Let $K=e^k$

$$V(t) = Ke^{-\frac{t}{RC}}$$

- At t=0 → $V(0) = e^0$ K = K = V_0

$$V(t) = V_0\, e^{-t/RC}$$

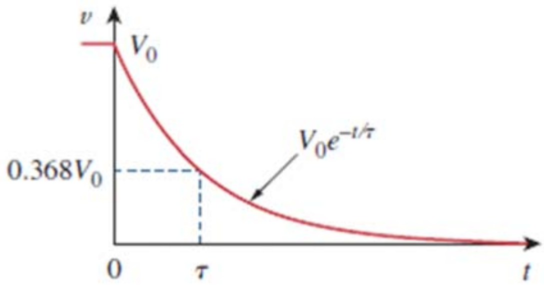
$$V(t) = V_0\, e^{-t/RC}$$



- voltage response of the *RC* circuit is an exponential decay of the initial voltage.
- natural response: response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source.
- the voltage decreases is expressed in terms of the *time constant*, denoted by τ "tau"

$$\tau = RC$$

$$v(t) = V_0 e^{-t/\tau}$$



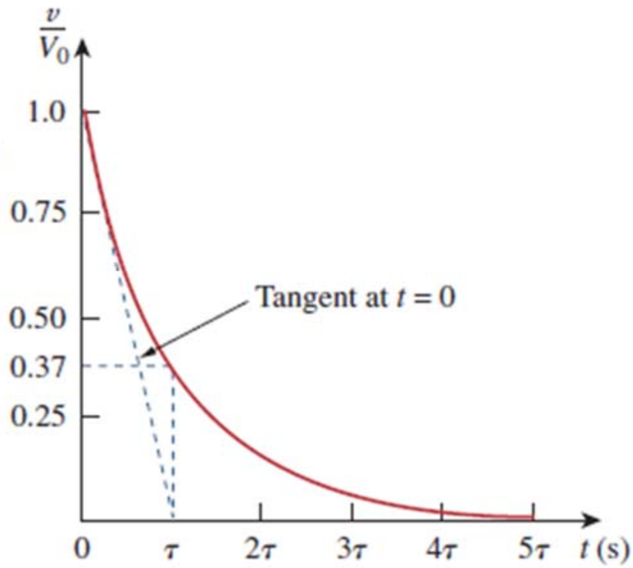
- The time constant of a circuit is the time required for the response to decay to a factor of 1/*e* or 36.8 % percent of its initial value.1

at $t = \tau$,

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674



- the voltage V(t) is less than 1 percent of after (five time τ constants)
- it takes (5 τ) for the circuit to reach its final state or steady state when no changes take place with time.

- smaller the time constant, the more rapidly the voltage decreases, the faster the response
- whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state.

- Calculating current $i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$
- power dissipated in the resistor is $p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$

- The energy absorbed by the resistor up to time *t* is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t \frac{V_0^2}{R} e^{-2\lambda/\tau} d\lambda$$

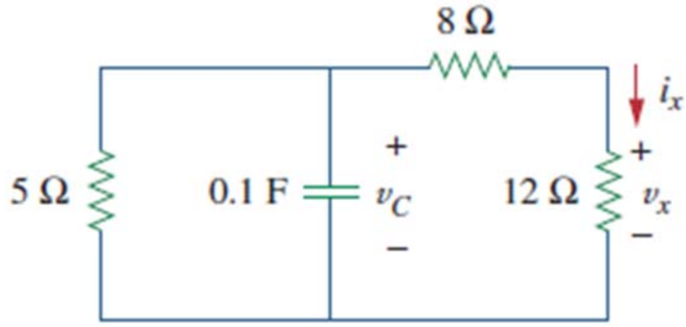
- From integration table

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

- $w_R(t) \equiv -\frac{\tau V_0^2}{2R} e^{-2\lambda/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = RC$
- Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$,
- The energy that was initially stored in the capacitor is eventually dissipated in the resistor

Example 01

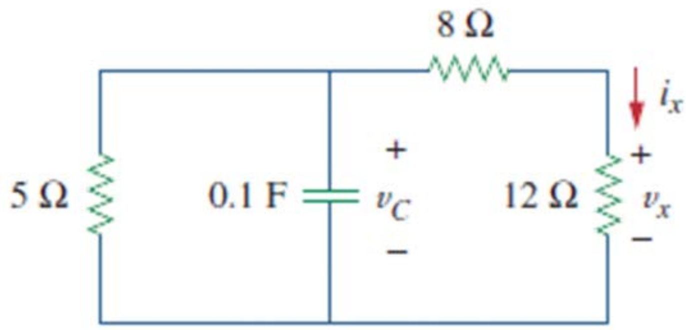
let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

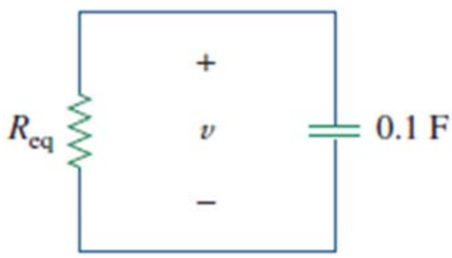


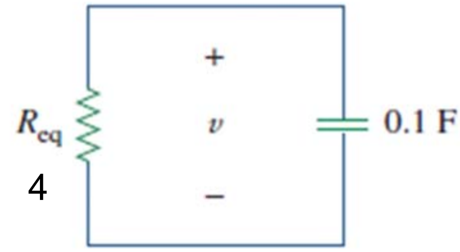
1. Our objective is always to first obtain capacitor voltage

- find the equivalent resistance or the Thevenin resistance at the capacitor terminals.
- 8 in series 12 → 20 ohm
- 20 in parallel 5 →

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

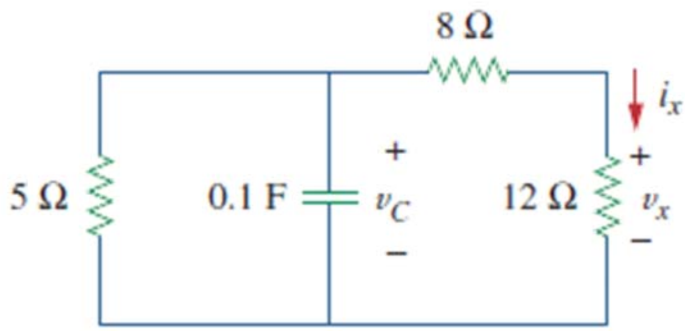






- The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$$



- Voltage response with time

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4} \text{ V},$$

$$v_C = v = 15e^{-2.5t} \text{ V}$$

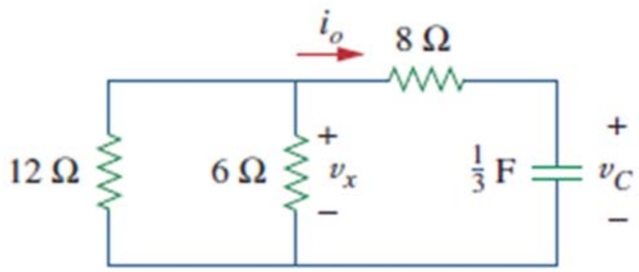
- Return to original circuit, we can use voltage division to get

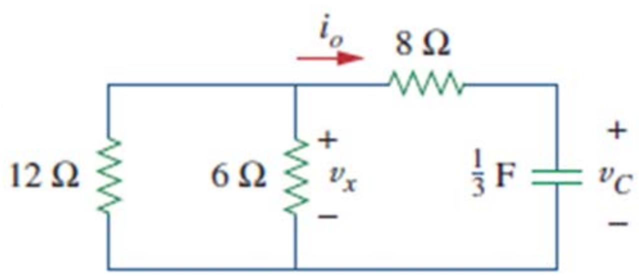
$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t} \text{ V}$$

- Finally, $$i_x = \frac{v_x}{12} = 0.75e^{-2.5t} \text{ A}$$

Example 02

Let $v_C(0) = 60$ V. Determine v_C, v_x, and i_o for $t \geq 0$.





- Find Req:

12 parallel with 6 → 4 ohm

4 series with 8 → 12 ohm

- Find time constant (τ)

τ = 12/3 = 4 sec

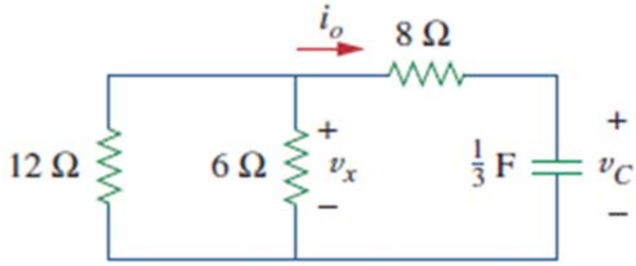
- Find Vc

$$v = v(0)e^{=-t/\tau} = 60\ e^{-0.25t}$$

- Return to original circuit and use VD

$$v_x = \frac{4}{4+8} Vc$$

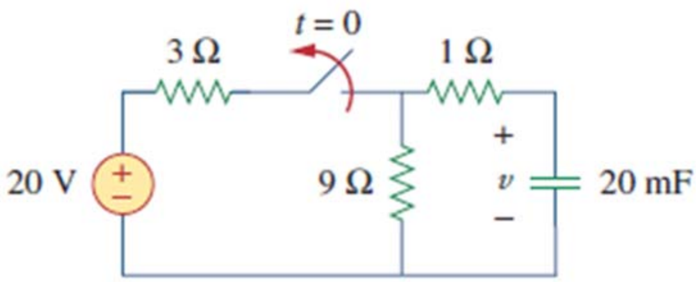
$$v_x = \frac{1}{3} \times 60e^{-0.25t} = 20\,e^{-0.25t}$$



$$I_o = -\frac{v_x}{4} = -5e^{-0.25t}$$

Example 03

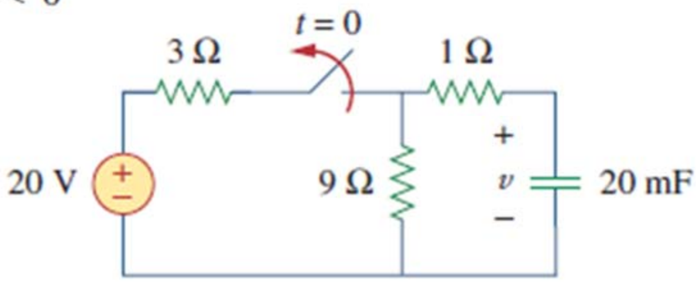
- The switch in the circuit in Fig. has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

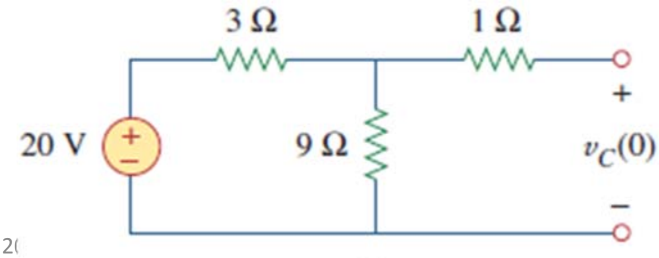


- For $t < 0$, the switch is closed; the capacitor is an open circuit to dc,
- Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

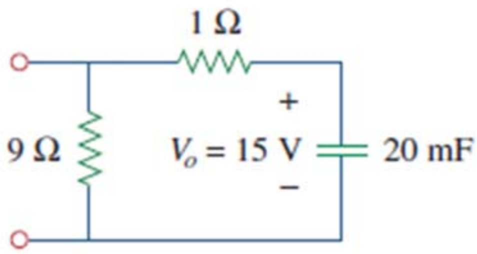
$$v_C(0) = V_0 = 15 \text{ V}$$





- For t>0 the switch is opened, and we have the *RC* circuit shown

$$R_{eq} = 1 + 9 = 10\ \Omega$$



- The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2 \text{ s}$$

- the voltage across the capacitor for t>0

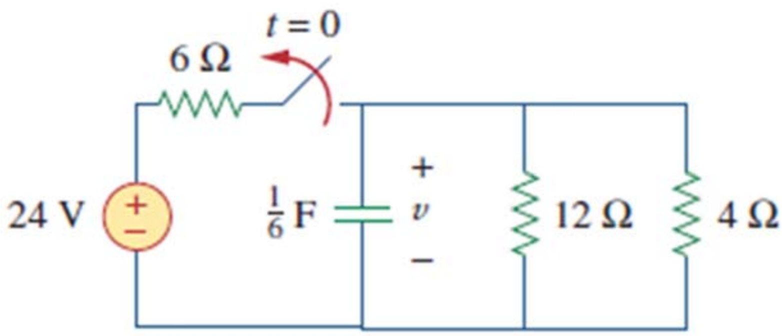
$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} \text{ V}$$

$$v(t) = 15e^{-5t} \text{ V}$$

- Initial energy $w_C(0) = \frac{1}{2}Cv_C^2 \quad = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25 \text{ J}$

Example 04

If the switch in Fig. opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

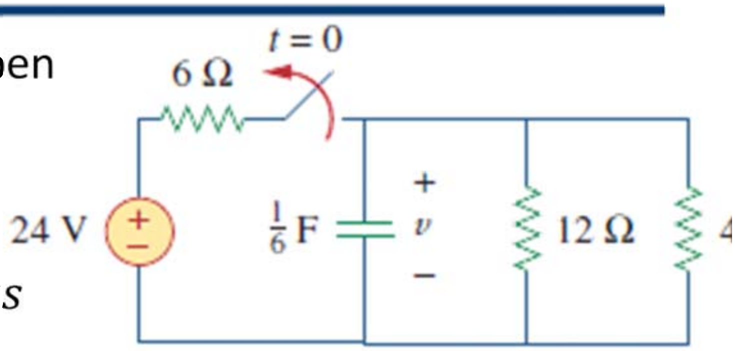


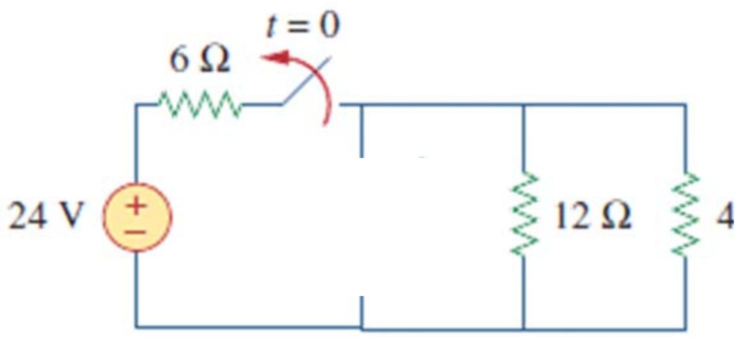
- At t<0 switch is close cap is open

12 in parallel with 4 → 3 ohm

- Using V.D.

$$Vc(0) = 24 \times \frac{3}{6+3} = 8\,Volts$$





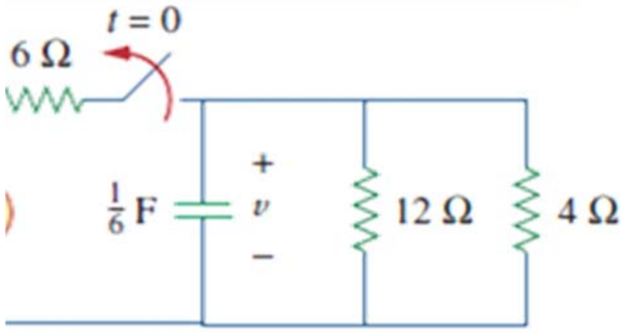
- For t>0, switch is open
- Req = 3 ohm
- Time constant

$$\tau = Req \times C = 3 \times \frac{1}{6} = 0.5\ s$$

$$v(t) = Vc(0)e^{-t/\tau} = 8e^{-2t}$$

$$Wc(0) = \frac{1}{2} C\ Vc^2 = \frac{1}{2}\frac{1}{6}\ 82$$

$$= 5.333\ w$$



Summery

<u>RC</u>

$$V = V0\ e^{-t/RC}$$

$$\tau = RC$$

$$V = V0\ e^{-t/\tau}$$

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

$$w_R(t) = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),$$

<u>RL</u>

$$I = I0e^{-tR/L}$$

$$\tau = \frac{L}{R}$$

$$I = I0e^{-t/\tau}$$

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

Thanks,..

See you next week (ISA),…