



Lecture (03) Methods of Analysis

By:

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Agenda

- Introduction
- Nodal analysis
- Nodal analysis with dependent source

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Cramer's Rule

Given a system of linear equations, Cramer's Rule is a handy way to solve for just one of the variables without having to solve the whole system of equations.

Let's use the following system of equations:

$$\begin{aligned}2v_1 + v_2 + v_3 &= 3 \\ v_1 - v_2 - v_3 &= 0 \\ v_1 + 2v_2 + v_3 &= 0\end{aligned}$$

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Cramer's Rule (2)

We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values.

Let Δ be the determinant of the coefficient matrix of the above system,

and

let Δ_x be the determinant formed by replacing the x -column values with the answer-column values:

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Cramer's Rule (3)

system of equations	coefficient matrix's determinant	answer column	D_x : coefficient determinant with answer-column values in x-column
$2v_1 + v_2 + v_3 = 3$ $v_1 - v_2 - v_3 = 0$ $v_1 + 2v_2 + v_3 = 0$	$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$	$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}$

Similarly, Δ_y and Δ_z would then be:

$$\Delta_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} \quad \Delta_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

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Cramer's Rule (4)

Evaluating each determinant, we get:

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 1(2+1) - -1(4-1) + 1(-2-1) = 3+3-3=3$$

$$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3(-1+2)=3$$

$$\Delta_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -3(1+1)=-6$$

$$\Delta_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 3(2+1)=9$$

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Cramer's Rule (5)

Cramer's Rule says that $x = \Delta_x \div \Delta$, $y = \Delta_y \div \Delta$, and $z = \Delta_z \div \Delta$.

That is:

$$x = 3/3 = 1, \quad y = -6/3 = -2, \quad \text{and} \quad z = 9/3 = 3$$

Done.

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Introduction

- nodal analysis, is based on a systematic application of Kirchhoff's current law (KCL),
- mesh analysis, is based on a systematic application of Kirchhoff's voltage law (KVL)
- You can analyze any linear circuit by obtaining a set of simultaneous equations that are then solved to obtain the required values of current or voltage.
- One method of solving simultaneous equations involves Cramer's rule,

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Nodal Analysis

1. Determine the non-reference nodes.
2. Define reference node, connect it to ground.
3. Define voltage at all other nodes, relative to reference node.
4. Apply KCL at these nodes, to find its reference voltage (assuming that all current leaving the node).
5. Transform back reference (nodal) voltages to the normal voltage drop.

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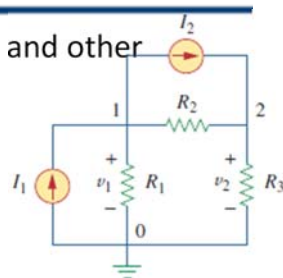
Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages v_1, v_2, \dots, v_{n-1} to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n - 1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

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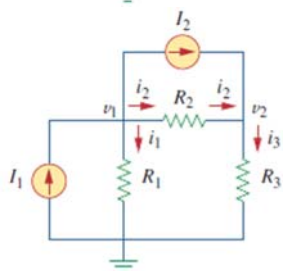
- The first step select reference node (ground), and other relative (nonreference) nodes



- applying KCL

@ 1 $I_1 = I_2 + i_1 + i_2$

@2 $I_2 + i_2 = i_3$



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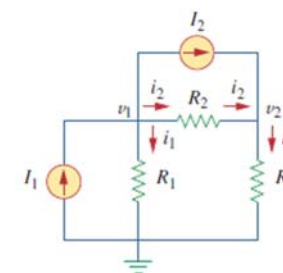
- apply Ohm's law to express the unknown currents

$$i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}$$

$$i_1 = \frac{v_1 - 0}{R_1}$$

$$i_2 = \frac{v_1 - v_2}{R_2}$$

$$i_3 = \frac{v_2 - 0}{R_3}$$



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Example 01

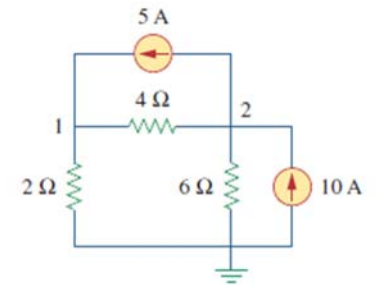
- Substituting

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

- Solving above equation together to get v_1 , and v_2

- Calculate the node voltages in the circuit shown in Fig.



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KCL@1

$$\frac{-v_1}{2} - \frac{v_1 - v_2}{4} + 5 = 0 \quad \times 4$$

$$-2v_1 - v_1 + v_2 + 20 = 0$$

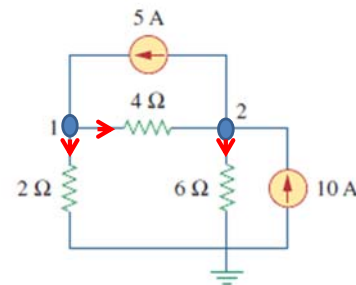
$$-3v_1 + v_2 = -20 \quad \rightarrow 1$$

KCL@2

$$\frac{v_1 - v_2}{4} - \frac{v_2}{6} + 10 - 5 = 0 \quad \times 12$$

$$3v_1 - 3v_2 - 2v_2 + 60 = 0$$

$$3v_1 - 5v_2 = -60 \quad \rightarrow 2$$



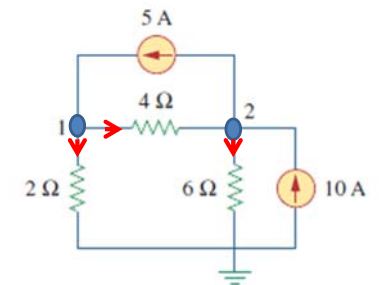
Add 1,2

$$-4v_2 = -80$$

$$v_2 = 20 \text{ volts}$$

Sub in 1

$$v_1 = \frac{+20 + v_2}{3} = \frac{+40}{3} = 13.33 \text{ volt}$$



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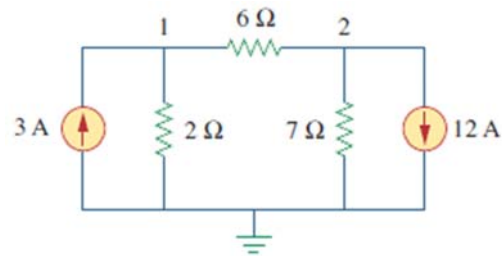
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Example 02

- Obtain the node voltages in the circuit



kcl@1

$$3 - \frac{v_1}{2} - \frac{v_1 - v_2}{6} = 0$$

$$-3v_1 - v_1 + v_2 = -18$$

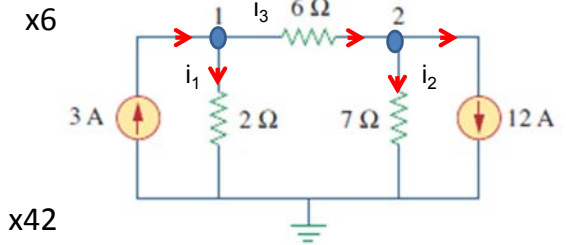
$$-4v_1 + v_2 = -18 \rightarrow 1$$

kcl@2

$$\frac{v_1 - v_2}{6} - \frac{v_2}{7} - 12 = 0$$

$$7v_1 - 7v_2 - 6v_2 = 504$$

$$7v_1 - 13v_2 = 504 \rightarrow 2$$



Mul 1 x 13 add to 2

$$-52v_1 + 13v_2 = -234$$

$$7v_1 - 13v_2 = 504$$

$$-45v_1 = 270$$

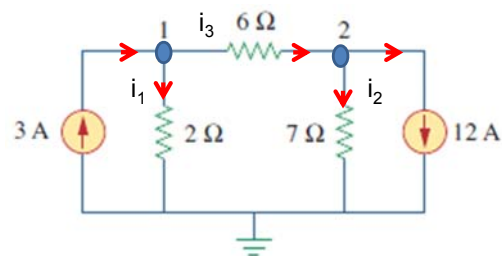
$$v_1 = 6 \text{ volt}$$

Sub @ 1

$$v_2 = -18 + 4(-6)$$

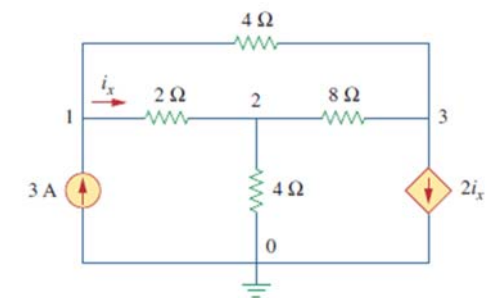
$$= -18 - 24$$

$$= -42 \text{ volt}$$



Example 03

- Determine the voltages at the nodes in Fig.



kcl@1

$$+3 - \frac{v1-v2}{2} - \frac{v1-v3}{4} = 0 \quad \times 4$$

$$12 - 2v1 + 2v2 - v1 + v3 = 0$$

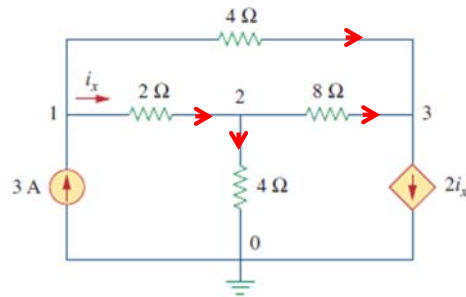
$$-3v1 + 2v2 + v3 = -12 \quad \rightarrow 1$$

kcl@2

$$\frac{v1-v2}{2} - \frac{v2}{4} - \frac{v2-v3}{8} = 0 \quad \times 8$$

$$4v1 - 4v2 - 2v2 - v2 + v3 = 0$$

$$4v1 - 7v2 + v3 = 0 \quad \rightarrow 3$$



Kcl@3

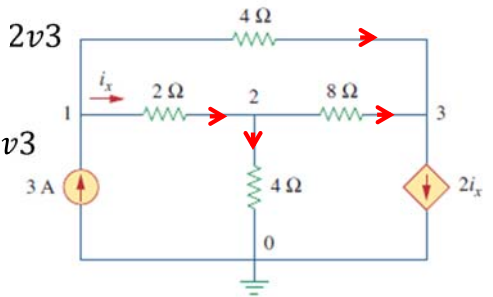
$$\frac{v2-v2}{8} - 2i_x + \frac{v1-v3}{4} = 0 \quad \times 8$$

$$v2 - v3 - 8\left(\frac{v1-v2}{2}\right) + 2v1 - 2v3 = 0$$

$$v2 - v3 - 8v1 + 8v2 + 2v1 - 2v3 = 0$$

$$-6v1 + 9v2 - 3v3 = 0 \quad /3$$

$$-2v1 + 3v2 - v3 = 0 \quad \rightarrow 3$$

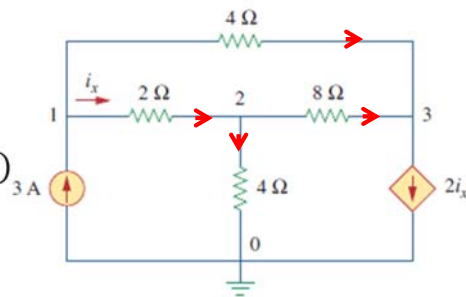


$$\begin{bmatrix} -3 & 2 & 1 \\ 4 & -7 & 1 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} -3 & 2 & 1 \\ 4 & -7 & 1 \\ -2 & 3 & -1 \end{bmatrix}$$

$$= 1(12 - 14) - 1(-9 + 4) - 1(21 - 8) = -10$$

$$\Delta 1 = \begin{bmatrix} 12 & 2 & 1 \\ 0 & -7 & 1 \\ 0 & 3 & -1 \end{bmatrix} = -12(7 - 3) = -48$$



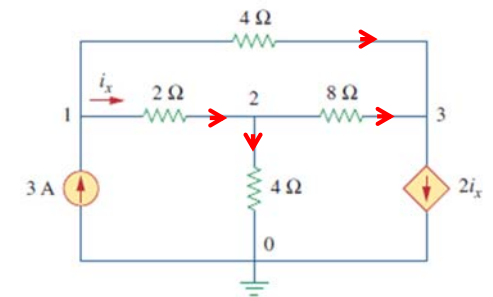
$$\Delta 2 = \begin{bmatrix} -3 & -12 & 1 \\ 4 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$= -12(-4 + 2) = -24$$

$$\Delta 3 = \begin{bmatrix} -3 & 2 & -12 \\ 4 & -7 & 0 \\ -2 & 3 & 0 \end{bmatrix}$$

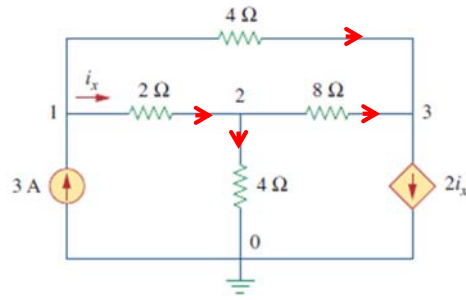
$$= -12(12 - 14) = 24$$

$$v1 = \frac{\Delta 1}{\Delta} = \frac{-48}{-10} = 4.8$$



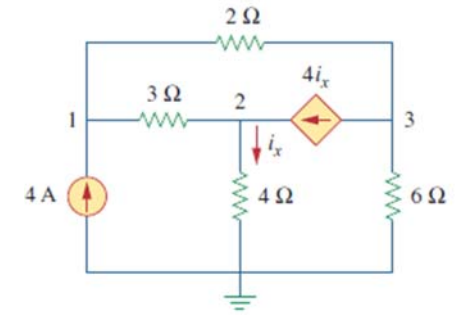
$$v_2 = \frac{\Delta_2}{\Delta} = \frac{-24}{-10} = 2.4$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{24}{-10} = -2.4$$



Example 04

- Find the voltages at the three nonreference nodes in the circuit of Fig.



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kcl@1

$$4 - \frac{v_1 - v_2}{3} - \frac{v_1 - v_3}{2} = 0 \quad \times 6$$

$$-2v_1 + 2v_2 - 3v_1 + 3v_3 = -24$$

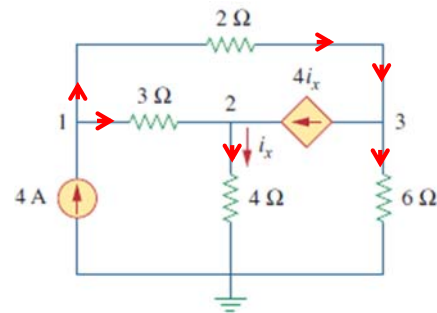
$$-5v_1 + 2v_2 + 3v_3 = -24 \quad \rightarrow 1$$

kcl@2

$$\frac{v_1 - v_2}{3} - \frac{v_2}{4} + 4\left(\frac{v_2}{4}\right) = 0 \quad \times 12$$

$$4v_1 - 4v_2 - 3v_2 + 12v_2 = 0$$

$$4v_1 - 5v_2 = 0 \quad \rightarrow 2$$



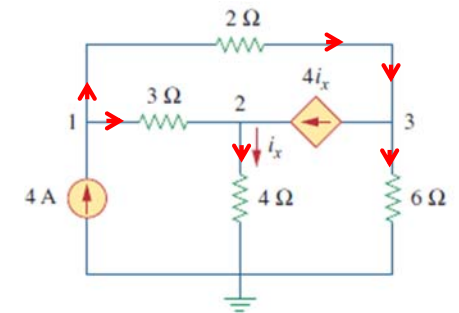
kcl@3

$$\frac{v_1 - v_3}{2} - 4\frac{v_2}{4} - \frac{v_3}{6} = 0 \quad \times 12$$

$$6v_1 - 6v_3 - 12v_2 - 2v_3 = 0$$

$$6v_1 - 12v_2 - 8v_3 = 0 \quad \rightarrow 3$$

$$\begin{pmatrix} -5 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & -12 & -8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} -24 \\ 0 \\ 0 \end{bmatrix}$$



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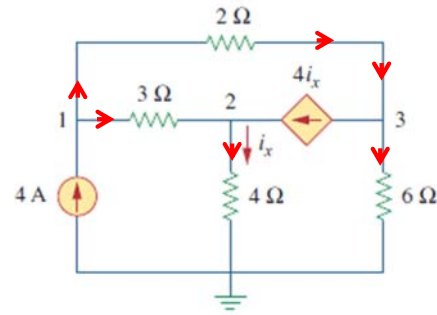
$$\Delta = \begin{pmatrix} -5 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & -12 & -8 \end{pmatrix}$$

$$= 3(-48 - 30) - 8(-25 - 8) = -30$$

$$\Delta 1 = \begin{pmatrix} -24 & 2 & 3 \\ 0 & 5 & 0 \\ 0 & -12 & -8 \end{pmatrix}$$

$$= -24(-40) = 960$$

$$V1 = \frac{\Delta 1}{\Delta} = 32 \text{ volt}$$



$$\Delta 2 = \begin{pmatrix} -5 & -24 & 3 \\ 4 & 0 & 0 \\ 6 & 0 & -8 \end{pmatrix}$$

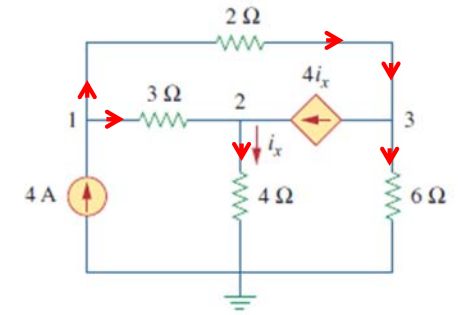
$$= 24(-32) = -768$$

$$V2 = -25.6 \text{ volt}$$

$$\Delta 3 = \begin{pmatrix} -5 & 2 & -24 \\ 4 & 5 & 0 \\ 6 & -12 & 0 \end{pmatrix}$$

$$= -24(-30 - 48) = -1872$$

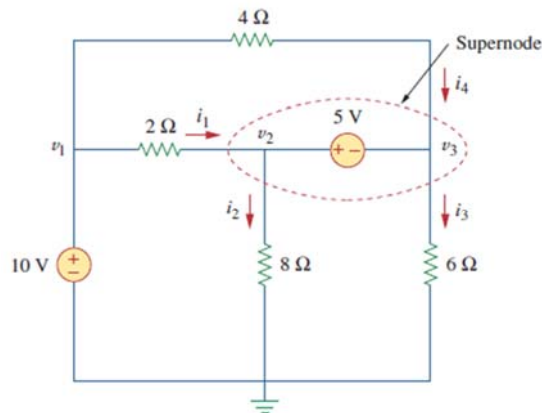
$$V3 = 62.4 \text{ volt}$$



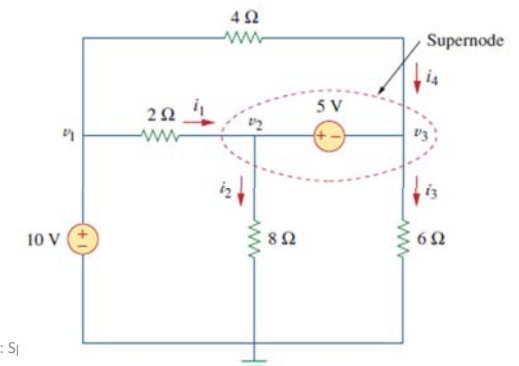
Nodal Analysis with Voltage Sources

- CASE 1** If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source.

$$v_1 = 10 \text{ V}$$



- CASE 2** If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.



- applying KCL

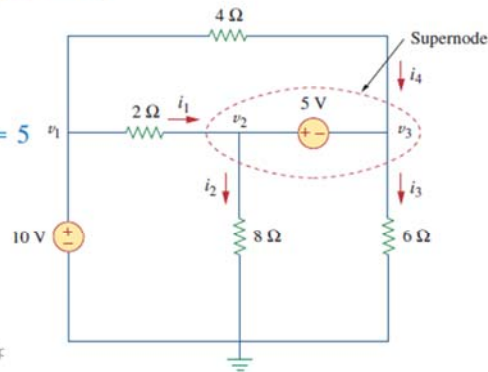
$$i_1 + i_4 = i_2 + i_3$$

- Ohm's law

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

- apply Kirchoff's voltage law

$$-v_2 + 5 + v_3 = 0 \Rightarrow v_2 - v_3 = 5$$



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- Super node rules:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

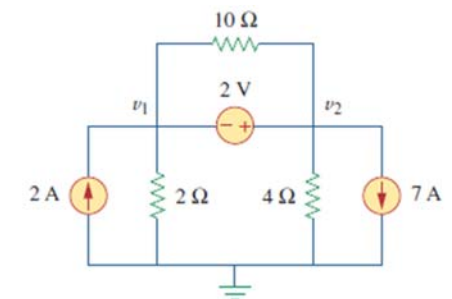
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- A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

Example 05

- For the circuit shown in Fig., find the node voltages.



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kcl@1

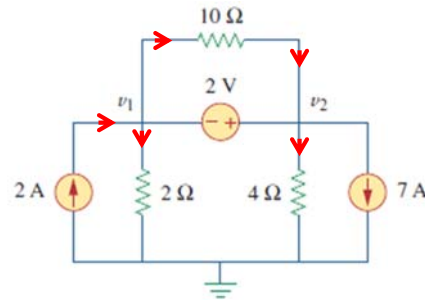
$$+2 - \frac{v_1}{2} - \frac{v_1 - v_2}{10} - 7 - \frac{v_2}{4}$$
$$+ - \frac{v_1 - v_2}{10} = 0$$

$$2 - \frac{v_1}{2} - \frac{v_2}{4} = 0 \times 4$$

$$2v_1 + v_2 = -20 \rightarrow 1$$

kvl@ super node 1,2

$$v_1 - v_2 = -2 \rightarrow 2$$



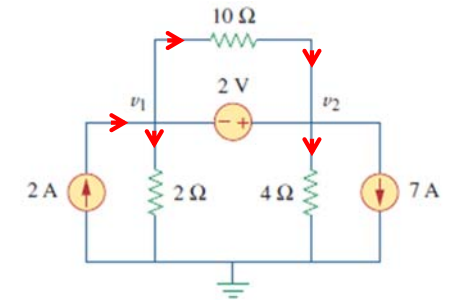
Add 1, 2

$$3v_1 = -22$$

$$v_1 = -7.33 \text{ volt}$$

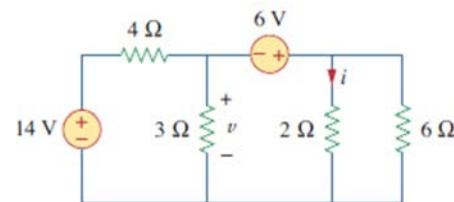
Sub in 2

$$v_2 = -7.33 + 2 = -5.33 \text{ volt}$$



Example 06

- Find v and i in the circuit of Fig



kcl@super1,2

$$\frac{14 - v_1}{4} - \frac{v_1}{3} - \frac{v_2}{2} - \frac{v_2}{6} = 0 \quad \times 12$$

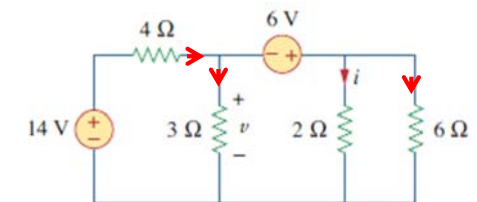
$$42 - 3v_1 - 4v_1 - 6v_2 - 2v_2 = 0$$

$$-7v_1 - 8v_2 = -42$$

$$7v_1 + 8v_2 = 42 \rightarrow 1$$

kvl@super1,2

$$v_2 - v_1 = 6 \rightarrow 2$$



Multiply 2 x 7 add to 1

$$-7v_1 + 7v_2 = 42$$

$$7v_1 + 8v_2 = 42$$

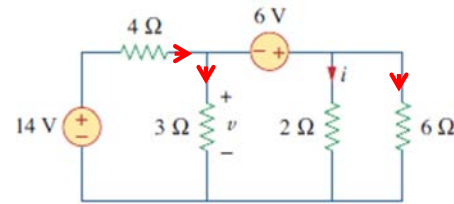
$$15v_2 = 84$$

$$v_2 = 5.6 \text{ volt}$$

$$i = \frac{v_2}{2} = 2.8 \text{ Amp}$$

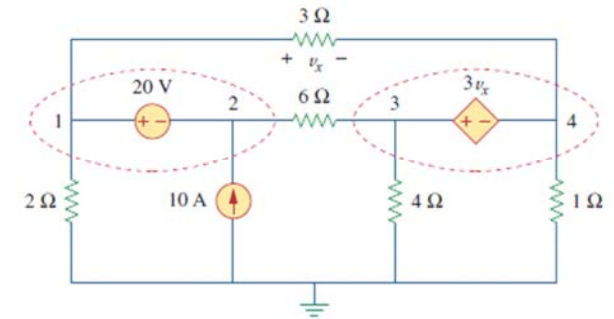
Sub in 2

$$v_1 = v_2 - 6 = -0.4 \text{ Amp}$$



Example 07

- Find the node voltages in the circuit of Fig



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kcl@ super 1,2

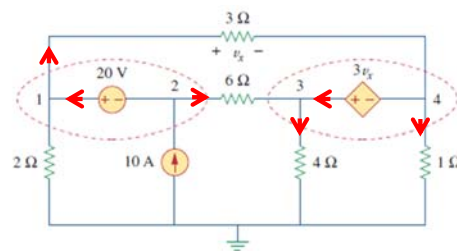
$$-\frac{v_1}{2} + 10 - \frac{v_1 - v_4}{3} - \frac{v_2 - v_3}{6}$$

$$= 0 \times 6$$

$$-3v_1 + 60 - 2v_1 + 2v_4 - v_2 + v_3 = 0$$

$$-5v_1 - v_2 + v_3 + 2v_4 = -60$$

$$\rightarrow 1$$



kvl@super 1,2

$$20 = v_1 - v_2$$

$$v_2 = v_1 - 20 \rightarrow 2$$

kcl@super 3,4

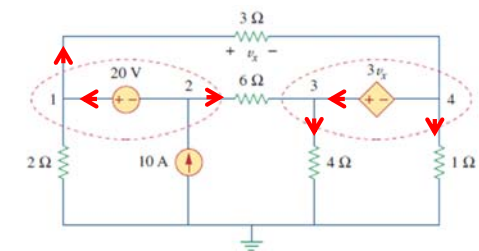
$$-\frac{v_4}{1} + \frac{v_1 - v_2}{3} - \frac{v_3}{4} + \frac{v_2 - v_3}{6} = 0$$

$$\times 12$$

$$-12v_4 + 4v_1 - 4v_3 - 3v_3 + 2v_2 - 2v_3 = 0$$

$$4v_1 + 2v_2 - 5v_3 - 12v_4 = 0$$

$$\rightarrow 3$$



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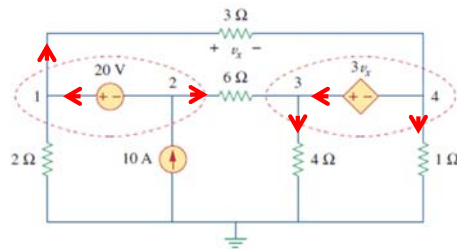
kvl@super 3,4

$$3v_x = v_3 - v_4$$

But $v_x = v_1 - v_4$

$$3v_1 - 3v_4 = v_3 - v_4$$

$$3v_1 - v_3 - 2v_4 = 0 \rightarrow 4$$



Sub 2 in 1,3

$$-5v_1 - v_1 + 20 + v_3 + 2v_4 = -60$$

$$-6v_1 + v_3 + 2v_4 = -80 \rightarrow 5$$

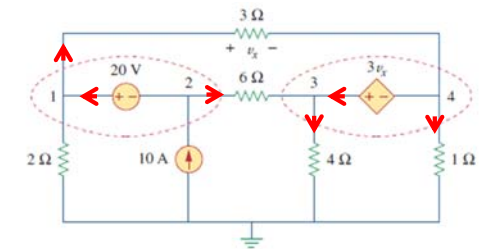
kvl@super 3,4

$$3v_x = v_3 - v_4$$

But $v_x = v_1 - v_4$

$$3v_1 - 3v_4 = v_3 - v_4$$

$$3v_1 - v_3 - 2v_4 = 0 \rightarrow 4$$



Sub 2 in 1

$$-5v_1 - v_1 + 20 + v_3 + 2v_4 = -60$$

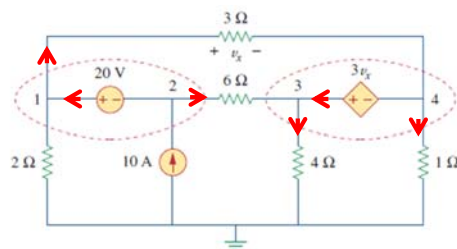
$$-6v_1 + v_3 + 2v_4 = -80 \rightarrow 5$$

Sub 2 in 3

$$+4v_1 + 2v_1 - 40 - 5v_3 - 16v_4 = 0$$

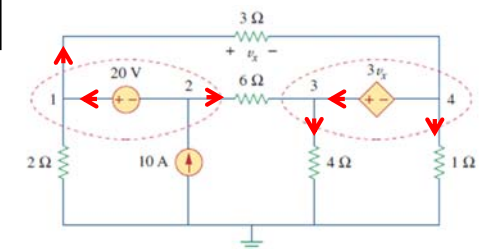
$$6v_1 - 5v_3 - 16v_4 = 40 \rightarrow 6$$

$$\begin{bmatrix} 3 & -1 & -2 \\ -6 & 1 & 2 \\ 6 & -5 & 16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -80 \\ 40 \end{bmatrix}$$



$$\begin{bmatrix} 3 & -1 & -2 \\ -6 & 1 & 2 \\ 6 & -5 & 16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -80 \\ 40 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 3 & -1 & -2 \\ -6 & 1 & 2 \\ 6 & -5 & 16 \end{bmatrix} = 1(96 - 12) + 1(-48 + 12) + 5(6 - 12) = 18$$

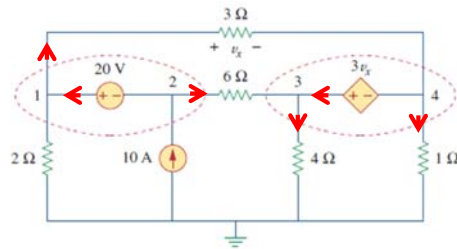


$$\Delta 1 = \begin{bmatrix} 0 & -1 & -2 \\ -80 & 1 & 2 \\ 40 & -5 & 16 \end{bmatrix}$$

$$= 80(16 - 10)$$

$$+ 40(-2 + 2) = 480$$

$$v_1 = \frac{\Delta 1}{\Delta} = \frac{480}{18} = 26.67 \text{ volt}$$



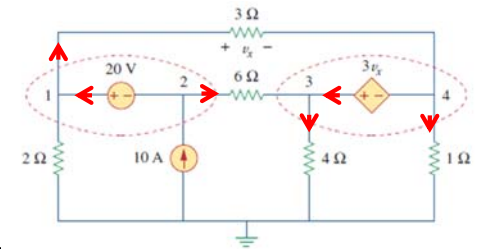
$$\Delta 2 = \begin{bmatrix} 3 & 0 & -2 \\ -6 & -80 & 2 \\ 6 & 40 & 16 \end{bmatrix}$$

$$= -80(-48 + 12)$$

$$- 40(6 - 12)$$

$$= 2880 + 240 = 3120$$

$$v_2 = \frac{\Delta 2}{\Delta} = \frac{3120}{18} = 173.33 \text{ volt}$$



$$\Delta 3 = \begin{bmatrix} 3 & -1 & 0 \\ -6 & 1 & -80 \\ 6 & -5 & 40 \end{bmatrix}$$

$$= 80(-15 + 6)$$

$$+ 40(3 - 6)$$

$$= -720 - 120 = -840$$

$$v_3 = \frac{\Delta 3}{\Delta} = \frac{-840}{18}$$

$$= -46.67 \text{ volt}$$

Sun in 2

$$v_2 = v_1 - 20 = 26.67 - 20$$

$$= 6.67 \text{ volt}$$

