Lecture (04) Electric Flux and Gauss's Law

By:

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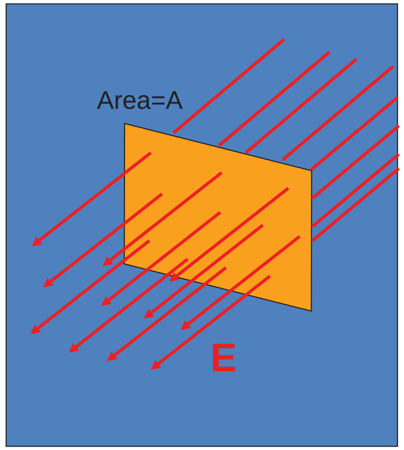
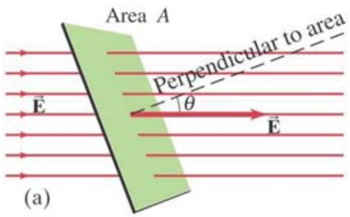
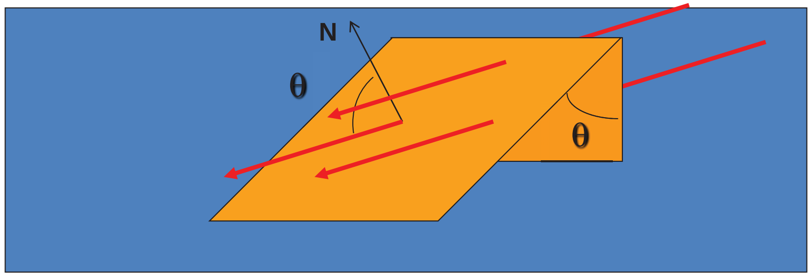
Electric Flux

- Electric flux is the number of Electric field lines penetrating a surface or an area.
- The number of field lines per unit of area is constant.
- The flux, Φ, is defined as the product of the field magnitude by the area crossed by the field lines.
- $\Phi = EA$



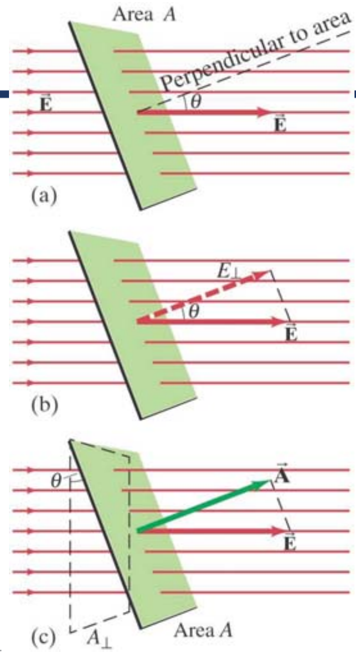
- If the surface is not perpendicular to the field, the expression of the field becomes:
- $\Phi = E\,A\,\cos\theta$
- Where θ is the angle between the field and a normal to the surface.





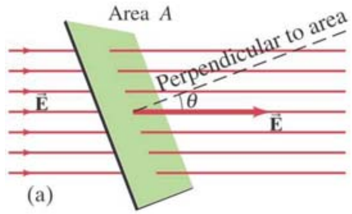
- (a) A uniform electric field **E** passing through a flat area *A*.
- (b) $E_\perp = E\cos\theta$ is the component of **E** perpendicular to the plane of area *A*.
- (c) $A_\perp = A\cos\theta$ is the projection (dashed) of the area *A* perpendicular to the field **E**.

$$[\Phi_E] = [E][A] = \left[\frac{N}{C}\right][m^2] \quad \text{as } E = \frac{F}{Q}$$

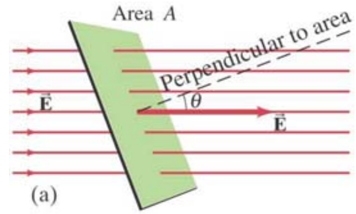


Example 01

- Calculate the electric flux through the rectangle shown. The rectangle is 10 cm by 20 cm, the electric field is uniform at 200 N/C, and the angle θ is 30°.



- $\Phi = E\ A\ \cos\theta$
- $\Phi = 200\ \times 0.2\ \times 0.1\ \times \cos 30 = 3.46\ \mathrm{N\cdot m^2/C}.$

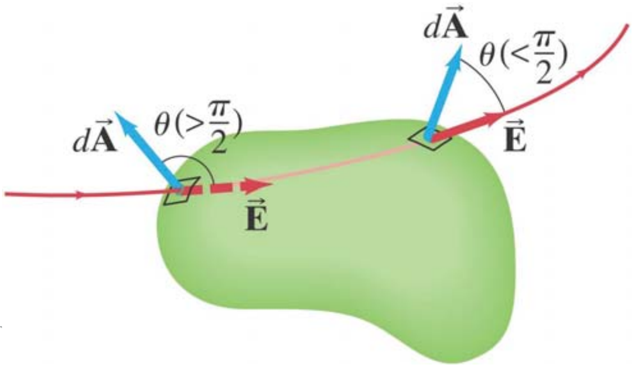


Gauss's Law

- The net number of field lines through the surface is proportional to the charge enclosed, and also to the flux,

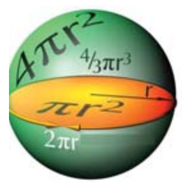
$$\Phi_E \approx \sum_{i=1}^{n} \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i.$$

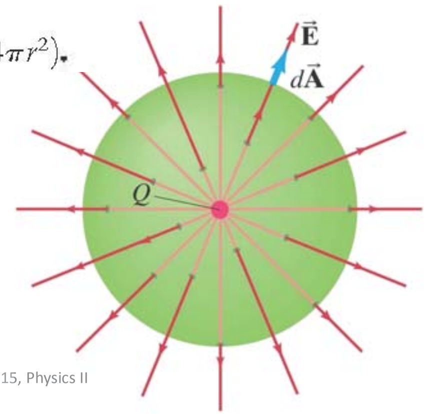
$$\Phi_E \approx \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}_i.$$

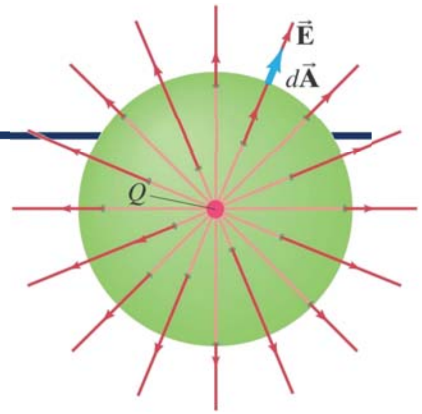


- A single point charge Q at the center of an imaginary sphere of radius r (our "gaussian surface"—that is, the closed surface we choose to use for applying Gauss's law in this case).
- For a point charge,

$$\Phi_E = \oint E\, dA = E\oint dA = E(4\pi r^2).$$





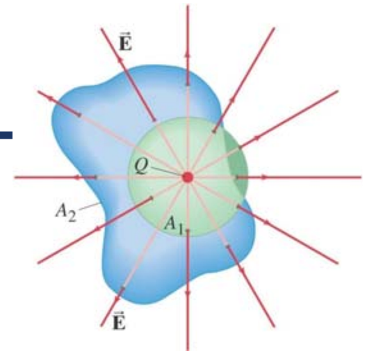


- *From columns low*
- $F = \frac{k\, q_1\, q_2}{r^2} = \frac{q_1\, q_2}{4\, \pi\, \varepsilon_0 r^2}$
- $But\ E = \frac{F}{Q}$
- $E = \frac{Q}{4\, \pi\, \varepsilon_0 r^2}$
- $\Phi = \oint E\, dA = E \oint dA$
- $= E\, 4\, \pi\, r^2 = \frac{k\, Q_{encl}}{r^2} 4\, \pi\, r^2 = \frac{Q_{encl}}{4\, \pi\, \varepsilon_0}\, 4\, \pi = \frac{Q_{encl}}{\varepsilon_0}$

- Using Coulomb's law to evaluate the integral of the field of a point charge over the surface of a sphere surrounding the charge gives:

$$\Phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$$

- Looking at the arbitrarily shaped surface A_2, we see that the same flux passes through it as passes through A_1. Therefore, this result should be valid for any closed surface.



A single point charge surrounded by a spherical surface, A_1, and an irregular surface, A_2.

- Finally, if a gaussian surface encloses several point charges, the superposition principle shows that:

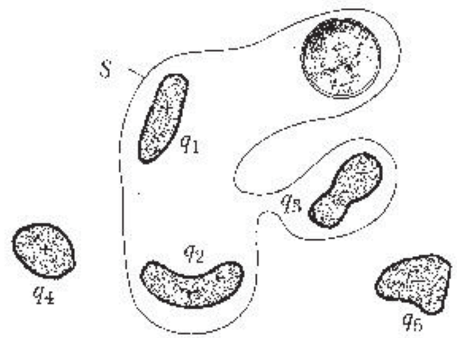
$$\Phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$$

- Therefore, Gauss's law is valid for any charge distribution.
- Note, however, that it only refers to the field due to charges within the gaussian surface – charges outside the surface will also create fields.

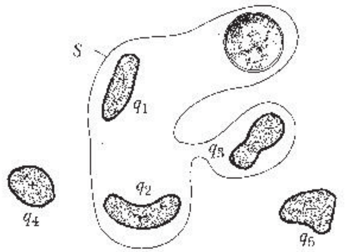
Example 02

- The figure shows five charged bodies of plastic and an electrically neutral coin as well as a Gaussian surface S. What is the flux through S if the charges are:?

$q_1 = q_4 = +3.1nC,\ q_2 = q_5 = -5.9nC,\ q_3 = -3.1nC$



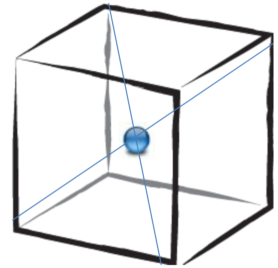
- $q_1 = q_4 = +3.1nC,\ q_2 = q_5 = -5.9nC,\ q_3 = -3.1nC$



$$\Phi = \oint_S E\,dA = \frac{q_{encl}}{\varepsilon_0}$$
$$= \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$
$$= \frac{(3.1 - 5.9 - 3.1) \times 10^{-9}}{8.55 \times 10^{-12}}$$
$$= -667\ N.m^2/C$$

Example 03

- A charge of 1.8 *nC* is placed at the center of a cube 3 *cm* on an edge. What is the electric flux through each face?



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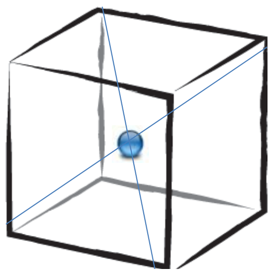
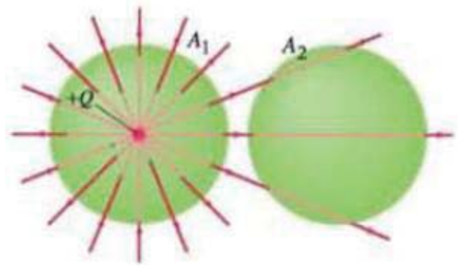
$$\Phi_{cube} = \oint_{cube} E\,dA = 6\oint_{Face} E\,dA = \frac{q_{encl}}{\varepsilon_0}$$



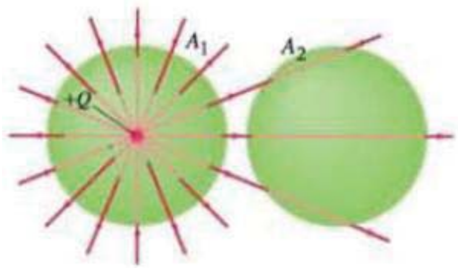
$$\oint_{Face} E\,dA = \frac{q_{encl}}{6\,\varepsilon_0}$$
$$= \frac{1.8 \times 10^{-9}}{6 \times 8.85 \times 10^{-12}}$$
$$= 34\ N.m^2/C$$

Example 04

- Consider the two gaussian surfaces, *A*1 and A2 , shown in Fig
- The only charge present is the charge *Q* at the center of surface A1
- What is the net flux through each surface, A1 and A2?

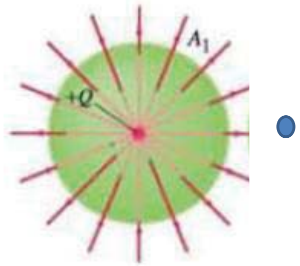


- the net flux through A1 is then
- $\Phi = \int E\, dA = \frac{Q_{encl}}{\varepsilon_0}$ = Q/ε_0
- Surface A2 encloses zero net charge, so the net electric flux through A2 is zero



MCQ

- A point charge *Q* is at the center of a spherical gaussian surface A. When a second charge *Q* is placed just outside *A* , the total flux through this spherical surface A is
- (a) unchanged,
- (b) doubled.
- (c) halved,
- (d) none of these.



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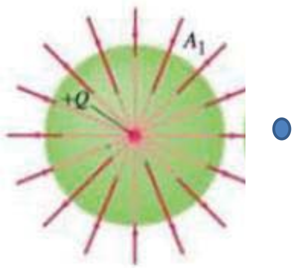
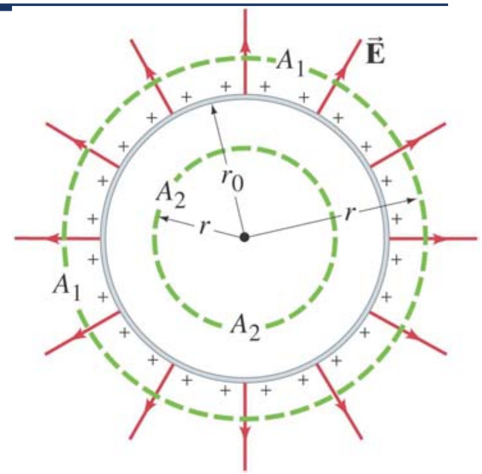
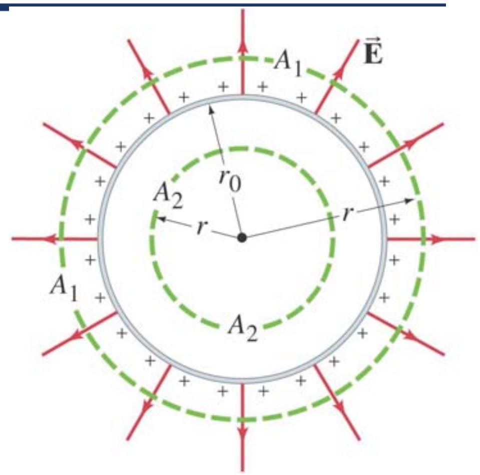


Example 05

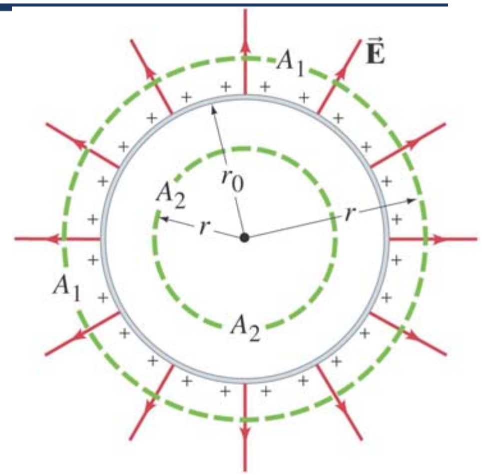
- Cross-sectional drawing of a thin spherical shell of radius r0 carrying a net charge Q uniformly distributed. A1 and A2 represent two gaussian surfaces
- Determine the electric field at points
- (a) outside the shell, and
- (b) within the shell.
- (c) What if the conductor were a solid sphere?



- a. The gaussian surface A_1, outside the shell, encloses the charge Q. We know the field must be radial, so
- $\Phi = \oint_{A_1} E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- $E\, 4\, \pi\, {r_1}^2 = \frac{Q_{encl}}{\varepsilon_0}$
- $E = \frac{Q_{encl}}{4\, \pi\, {r_1}^2 \varepsilon_0}$
- b. The gaussian surface A_2, inside the shell, encloses no charge; therefore the field must be zero.

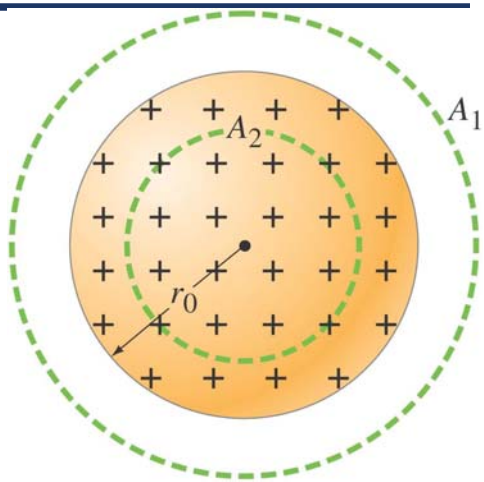


- c. All the excess charge on a conductor resides on its surface, so these answers hold for a solid sphere as well.
- $E = \frac{Q_{encl}}{4\, \pi\, {r_0}^2 \varepsilon_0}$

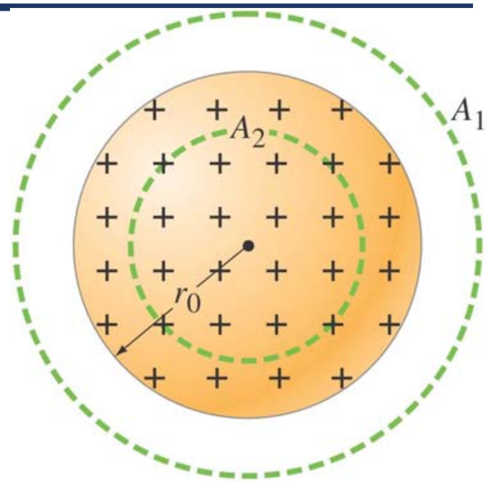


Example 06

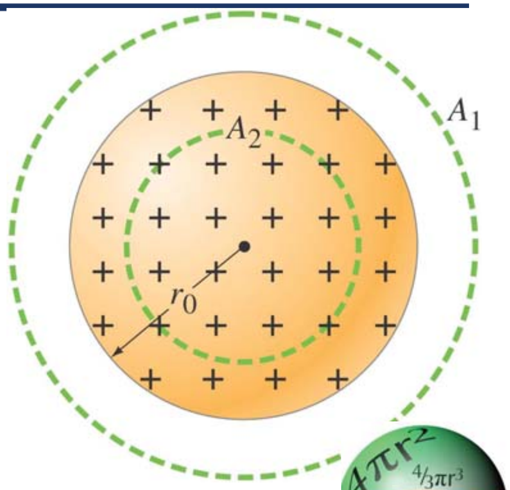
- Solid sphere of charge.
- An electric charge Q is distributed uniformly throughout a nonconducting sphere of radius r_0.
- Determine the electric field
- (a) outside the sphere ($r > r_0$) and
- (b) inside the sphere ($r < r_0$).



- a. Outside the sphere, a gaussian surface encloses the total charge Q. Therefore,
- $\Phi = \oint_{A_1} E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- $E\, 4\, \pi\, r^2 = \frac{Q_{encl}}{\varepsilon_0}$
- $E = \frac{Q_{encl}}{4\, \pi\, r^2 \varepsilon_0}$

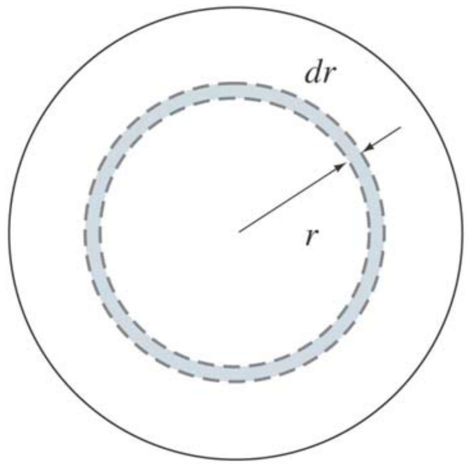


- b. Within the sphere, a spherical gaussian surface encloses a fraction of the charge
- $\phi = \oint_{A_2} E\, dA = \frac{Q_{encl\,A_2}}{\varepsilon_0}$
- $E\, 4\,\pi\, r^2 = \frac{Q_{encl\,A_2}}{\varepsilon_0}$
- $\rho = \frac{dQ}{dV}$
- $Q = \int_0^r \rho\, dV = \rho\, {}^4/_3\, \pi\, r_0{}^3$
- $Q_{encl\,A_2} = \int_0^{r_0} \rho\, dV = \rho\, {}^4/_3\, \pi\, r^3$
- $Q_{encl\,A_2} = \mathrm{Q}\, \frac{r^3}{r_0{}^3}$
- $E = \frac{Q\, r^3}{4\,\pi\, r^2\, \varepsilon_0\, r_0{}^3} = \frac{Q\, r}{4\,\pi\, \varepsilon_0 r_0{}^3}$

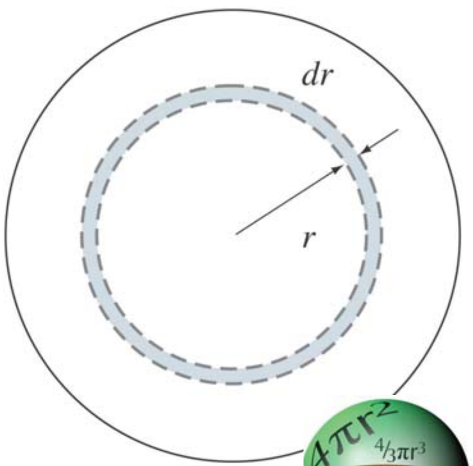


Example 07

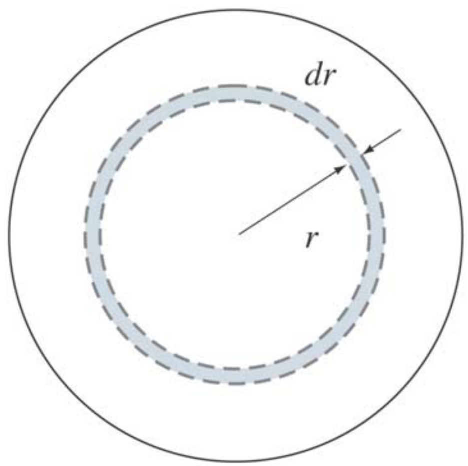
- Nonuniformly charged solid sphere.
- Suppose the charge density of a solid sphere is given by $\rho_E = \alpha\, r^2$, where α is a constant.
- (a) Find α in terms of the total charge Q on the sphere and its radius r_0.
- (b) Find the electric field as a function of r inside the sphere.



- a. Consider the sphere to be made of a series of spherical shells, each of radius r and thickness dr. The volume of each is
- $dV = 4\,\pi\, r^2$dr
- To find the total charge:
- $dQ = \rho\, dV = 4\,\pi\, \rho\, r^2\, dr$
- $dQ = 4\,\pi\, \alpha\, r^4\, dr$
- $Q = 4\,\pi\, \alpha\, \int_0^{r_0} r^4\, dr$
  $= \frac{4}{5}\pi\, \alpha\, r_0{}^5$
- $\alpha = \frac{5}{4\,\pi\, Q\, r_0{}^5}$

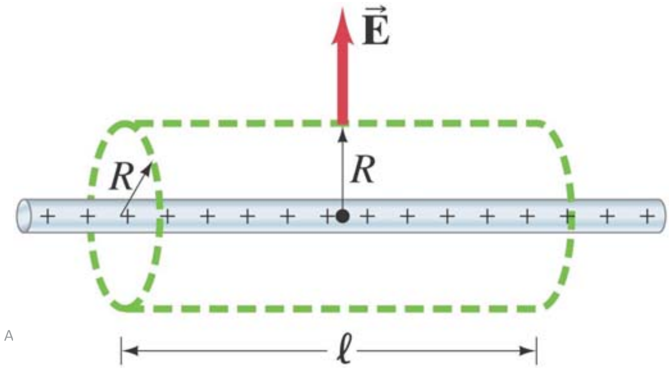


- $as\ Q = \frac{4}{5}\pi\, \alpha\, r_0{}^5$
- $as\!: Q_{encl} = \frac{4}{5}\pi\, \alpha\, r^5$
- $Q_{encl} = \frac{r^5}{r_0{}^5}\, Q$
- $\phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- $E\, 4\pi\, r^2 = \frac{r^5}{r_0{}^5\, \varepsilon_0}\, Q$
- $E = \frac{Q\, r^3}{4\,\pi\, r_0{}^5\, \varepsilon_0}$

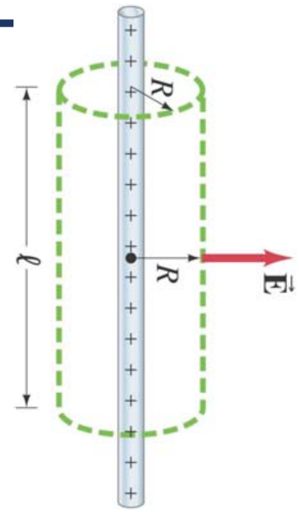


Example 08

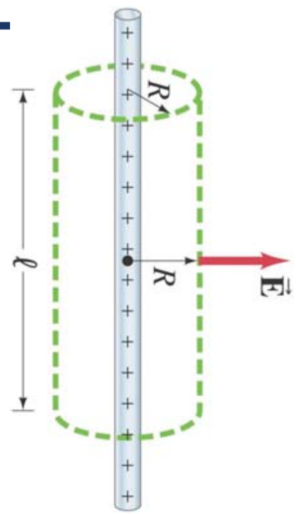
- Long uniform line of charge.
- A very long straight wire possesses a uniform positive charge per unit length, λ. Calculate the electric field at points near (but outside) the wire, far from the ends.



- If the wire is essentially infinite, it has cylindrical symmetry and we expect the field to be perpendicular to the wire everywhere
- Therefore, a cylindrical gaussian surface will allow the easiest calculation of the field.
- The field is parallel to the ends and constant over the curved surface; integrating over the curved surface

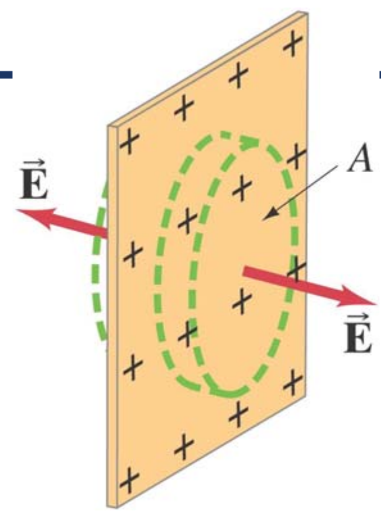


- $\Phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- For a cylinder $A = 2\, \pi\, R\, l$
- And $Q_{encl} = \lambda\, l$
- $E\, 2\, \pi\, R\, l = \frac{\lambda\, l}{\varepsilon_0}$
- $E = \frac{\lambda}{2\, \pi\, R\, \varepsilon_0}$

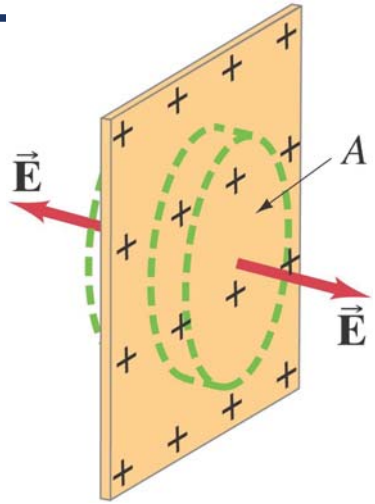


Example 09

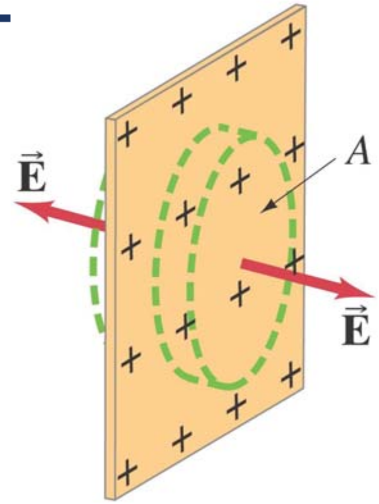
- Infinite plane of charge.
- Charge is distributed uniformly, with a surface charge density σ (σ = charge per unit area = dQ/dA) over a very large but very thin nonconducting flat plane surface.
- Determine the electric field at points near the plane.



- We expect E to be perpendicular to the plane, and choose a cylindrical gaussian surface with its flat sides parallel to the plane.
- The field is parallel to the curved side; integrating over the flat sides

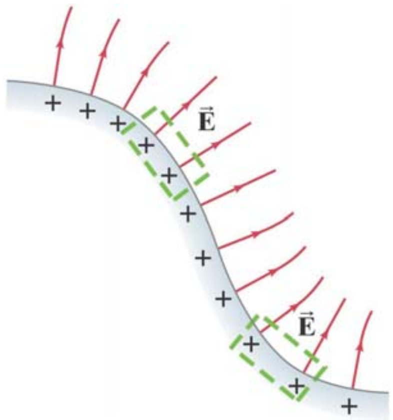


- $\Phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- $Q_{encl} = \sigma A$
- $2\, E\, A = \frac{\sigma A}{\varepsilon_0}$
- $E = \frac{\sigma}{2\, \varepsilon_0}$

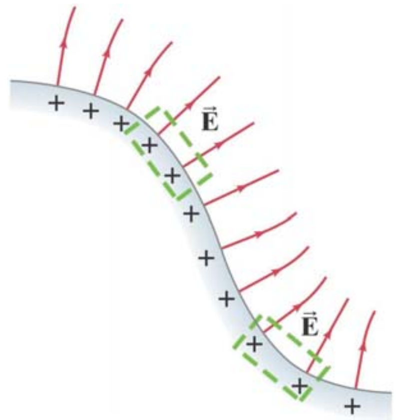


Example 10

- Electric field near any conducting surface.
- Show that the electric field just outside the surface of any good conductor of arbitrary shape is given by
- $E = \sigma/\varepsilon_0$
- where σ is the surface charge density on the conductor's surface at that point.

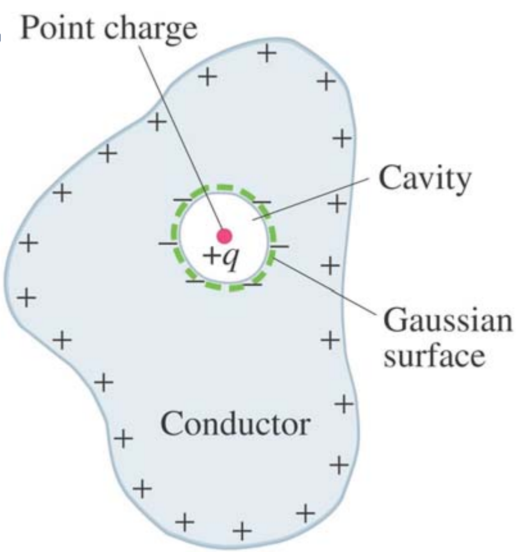


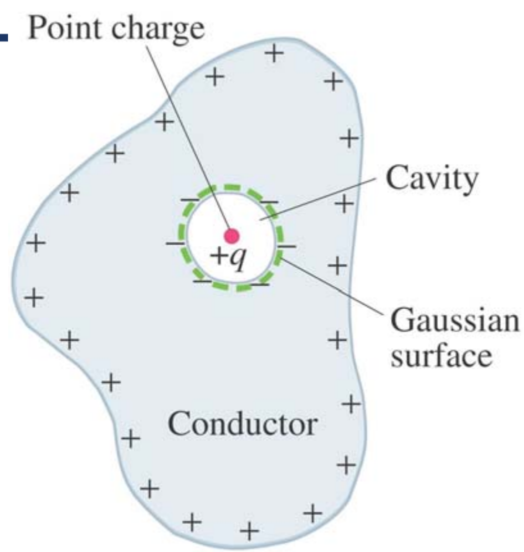
- Again we choose a cylindrical gaussian surface.
- Now, however, the field inside the conductor is zero, so we only have a nonzero integral over one surface of the cylinder.
- $\Phi = \oint E\, dA = \frac{Q_{encl}}{\varepsilon_0}$
- $E\, A = \frac{\sigma A}{\varepsilon_0}$
- $E = \frac{\sigma}{\varepsilon_0}$



Example 11

- Conductor with charge inside a cavity.
- Suppose a conductor carries a net charge $+Q$ and contains a cavity, inside of which resides a point charge $+q$.
- What can you say about the charges on the inner and outer surfaces of the conductor?





- The field must be zero within the conductor, so the inner surface of the cavity must have an induced charge totaling $-q$ (so that a gaussian surface just around the cavity encloses no charge).
- The charge $+Q$ resides on the outer surface of the conductor.

- **Electric flux:** $\Phi_E = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}.$
- **Gauss's law:** $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{encl}}{\epsilon_0}.$
- **Gauss's law can be used to calculate the field in situations with a high degree of symmetry.**
- **Gauss's law applies in all situations.**